ECE 3640 - Discrete-Time Signals and Systems

z-transform

Jake Gunther

Spring 2015

UtahStateUniversity

Department of Electrical & Computer Engineering

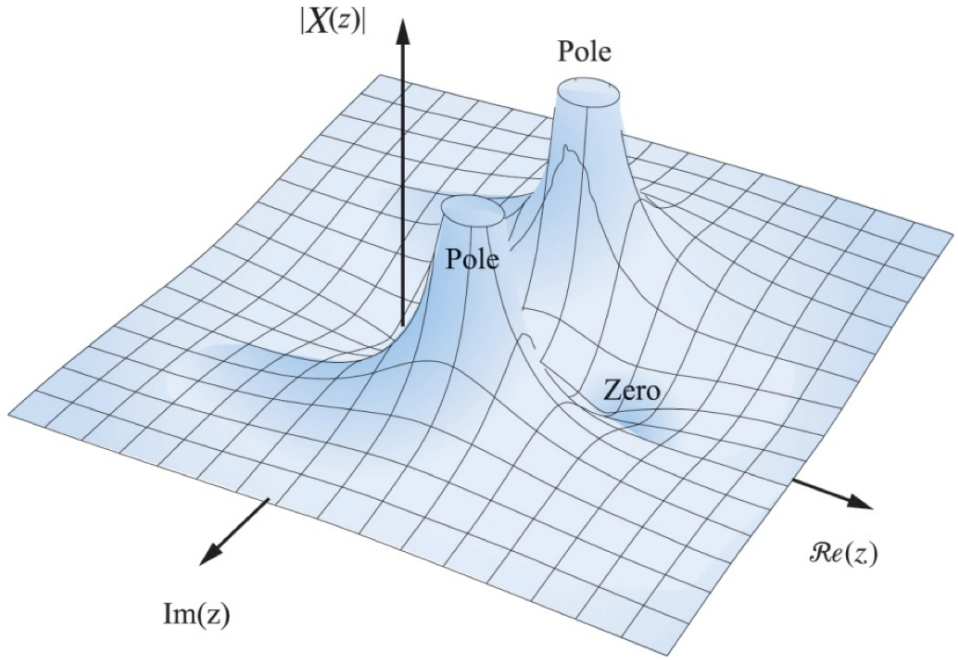
z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\text{ROC} = \{z \in \mathbb{C} : |X(z)| < \infty\}$$

- need to specify both algebraic formula for $X(z)$ and the ROC
- the ROC never contains poles
- zeros may lie in the ROC
- ROC is always circular

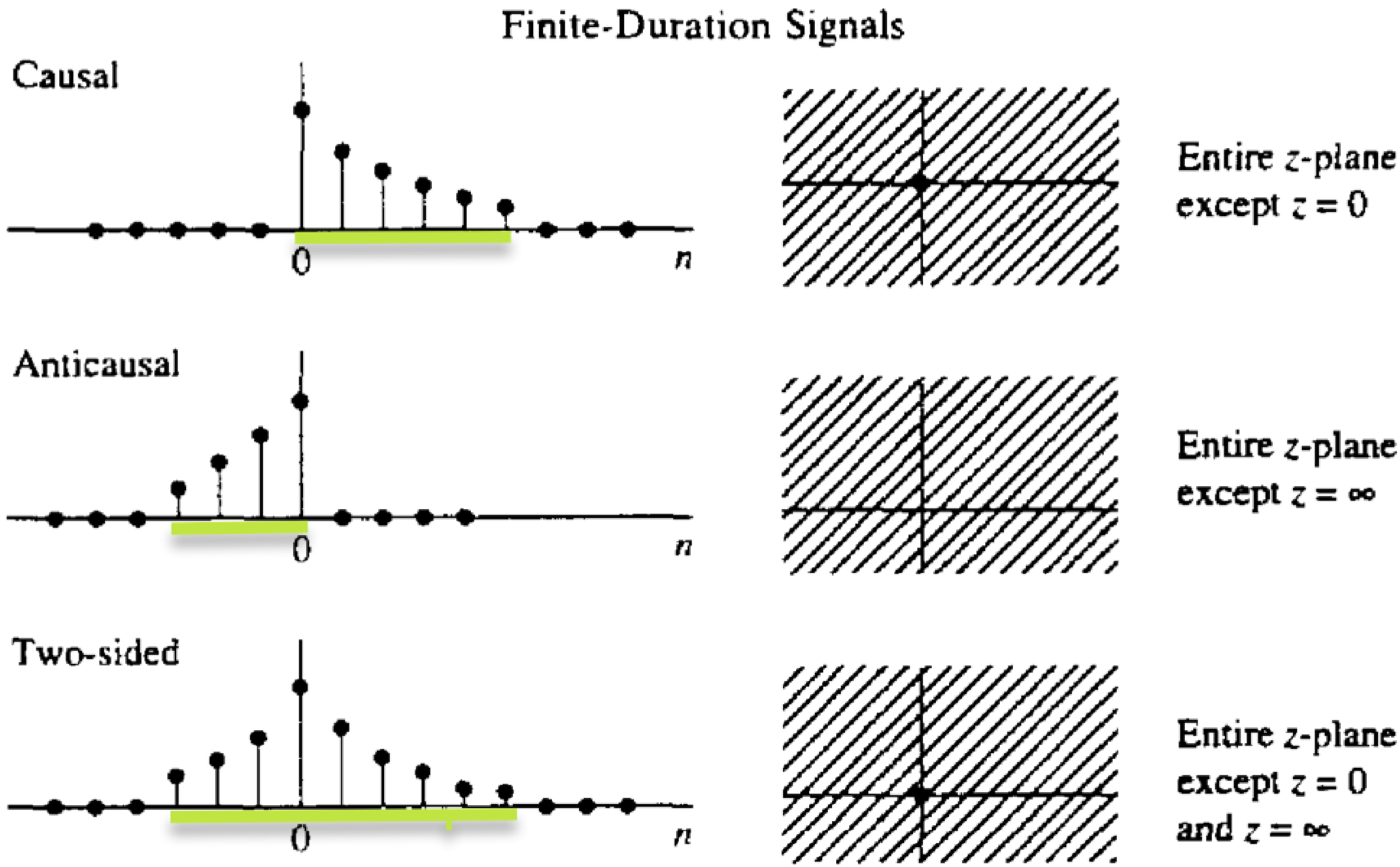
z-transform is a "surface"



This figure was taken from Manolakis & Ingle "Applied Digital Signal Processing,", Cambridge University Press.

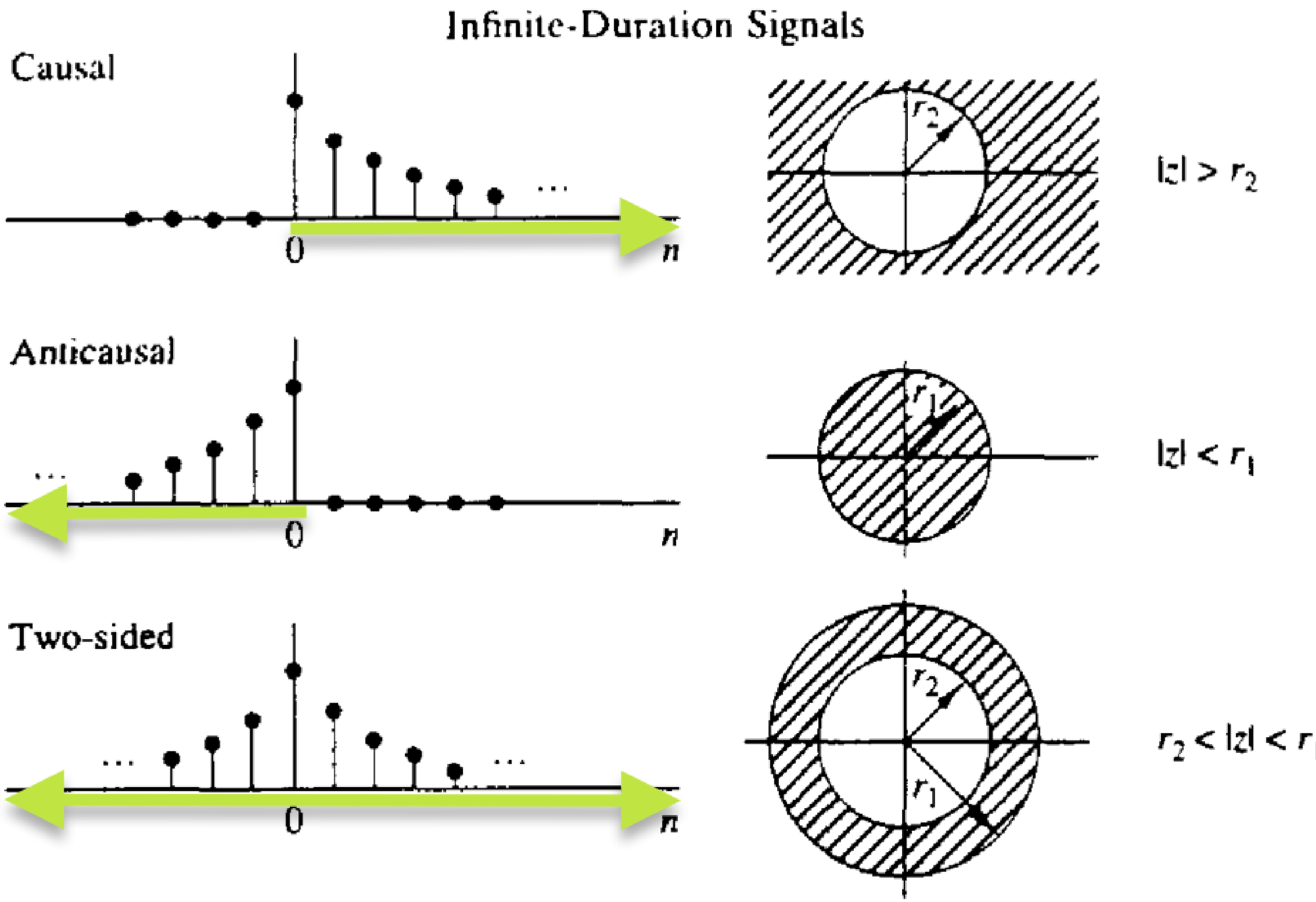
- actually $|X(z)|, \angle X(z), \Re\{X(z)\}, \Im\{X(z)\}$ are surfaces
- for poles $|X(z)| \to \infty$
- for zeros $|X(z)| = 0$

finite duration signals



This figure was taken from Proakis & Manolakis "Digital Signal Processing: Principles, Algorithms and Applications,", 3rd edition, Prentice Hall.

infinite duration signals



This figure was taken from Proakis & Manolakis "Digital Signal Processing: Principles, Algorithms and Applications,", 3rd edition, Prentice Hall.

example

$$x[n] = a^n u[n]$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

pole: $z = a$

zero: $z = 0$

ROC: $|z| > |a|$

(convergence outside a circle)

(causal (right-sided) signal)

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

pole: $z = a$

zero: $z = 0$

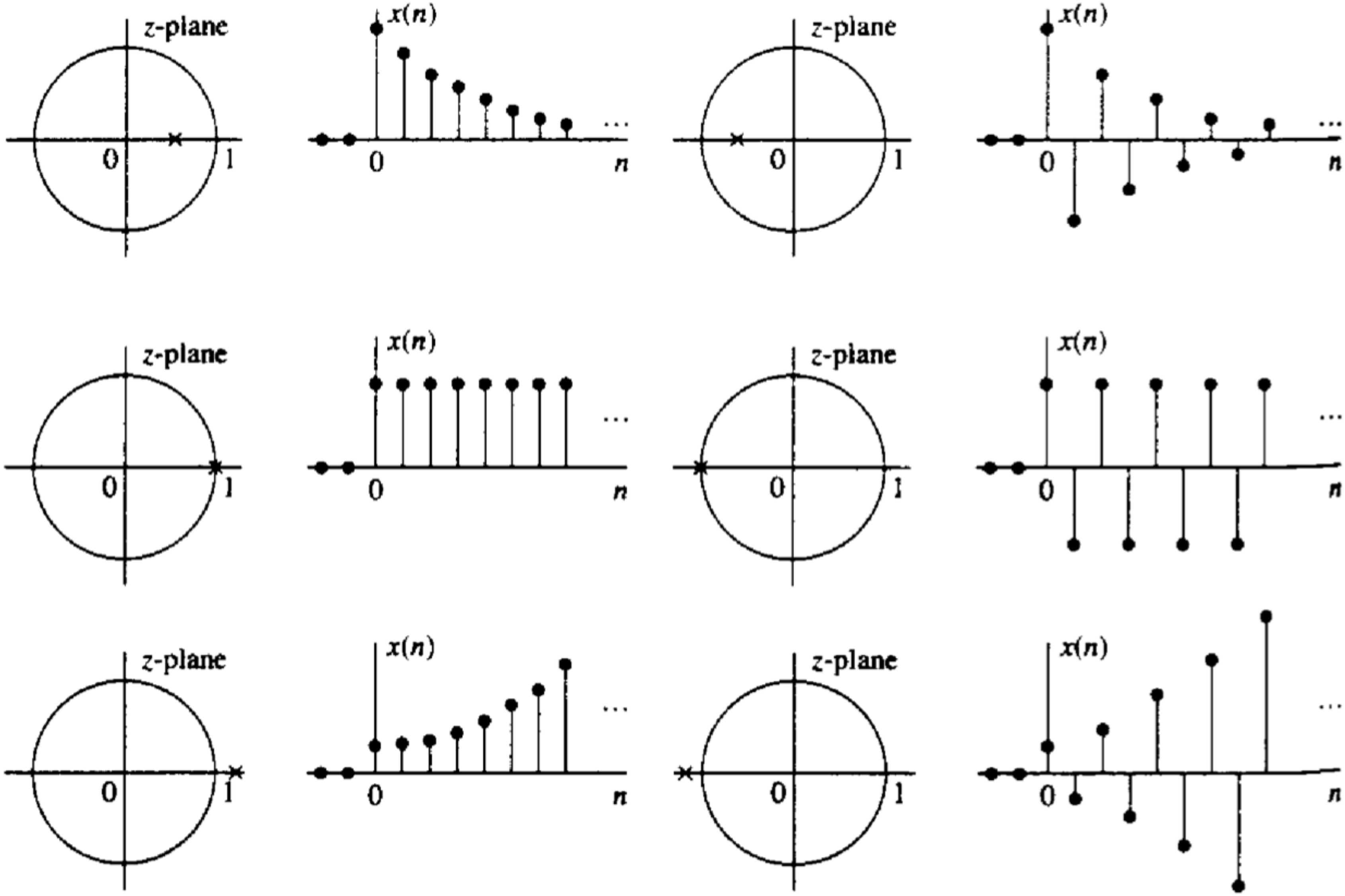
ROC: $|z| < |a|$

(convergence inside a circle)

(anticausal (left-sided) signal)

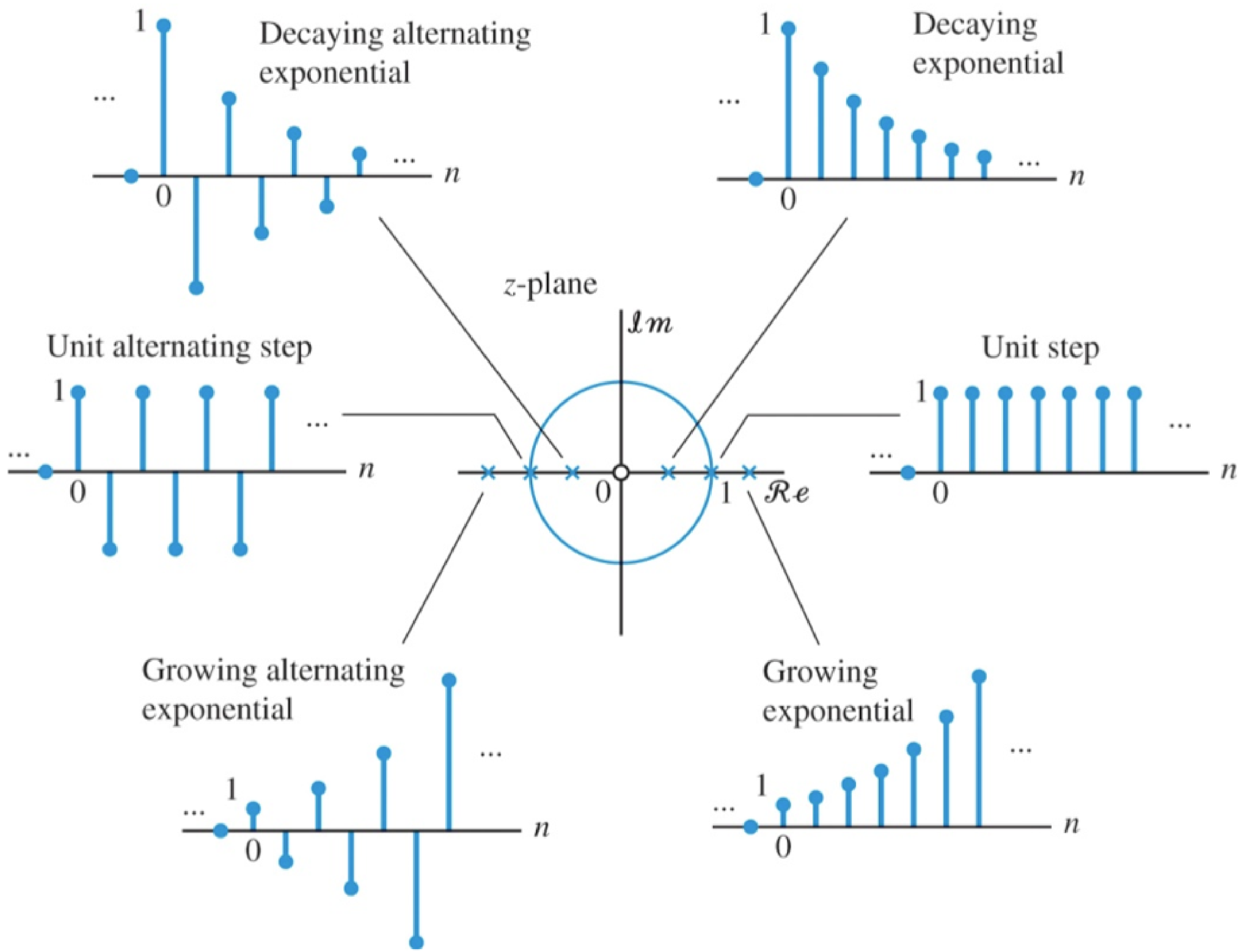
- two different signals can give the same formula for $X(z)$
- they will have different ROCs

first order poles



This figure was taken from Proakis & Manolakis “Digital Signal Processing: Principles, Algorithms and Applications,”, 3rd edition, Prentice Hall.

signal and pole examples



This figure was taken from Manolakis & Ingle "Applied Digital Signal Processing,", Cambridge University Press.

example

$$x[n] = na^n u[n]$$

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}$$

pole: $z = a$ (2nd order pole)

zero: $z = 0$

ROC: $|z| > |a|$

(convergence outside a circle)

(causal (right-sided) signal)

$$x[n] = -na^n u[-n-1]$$

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}$$

pole: $z = a$ (2nd order pole)

zero: $z = 0$

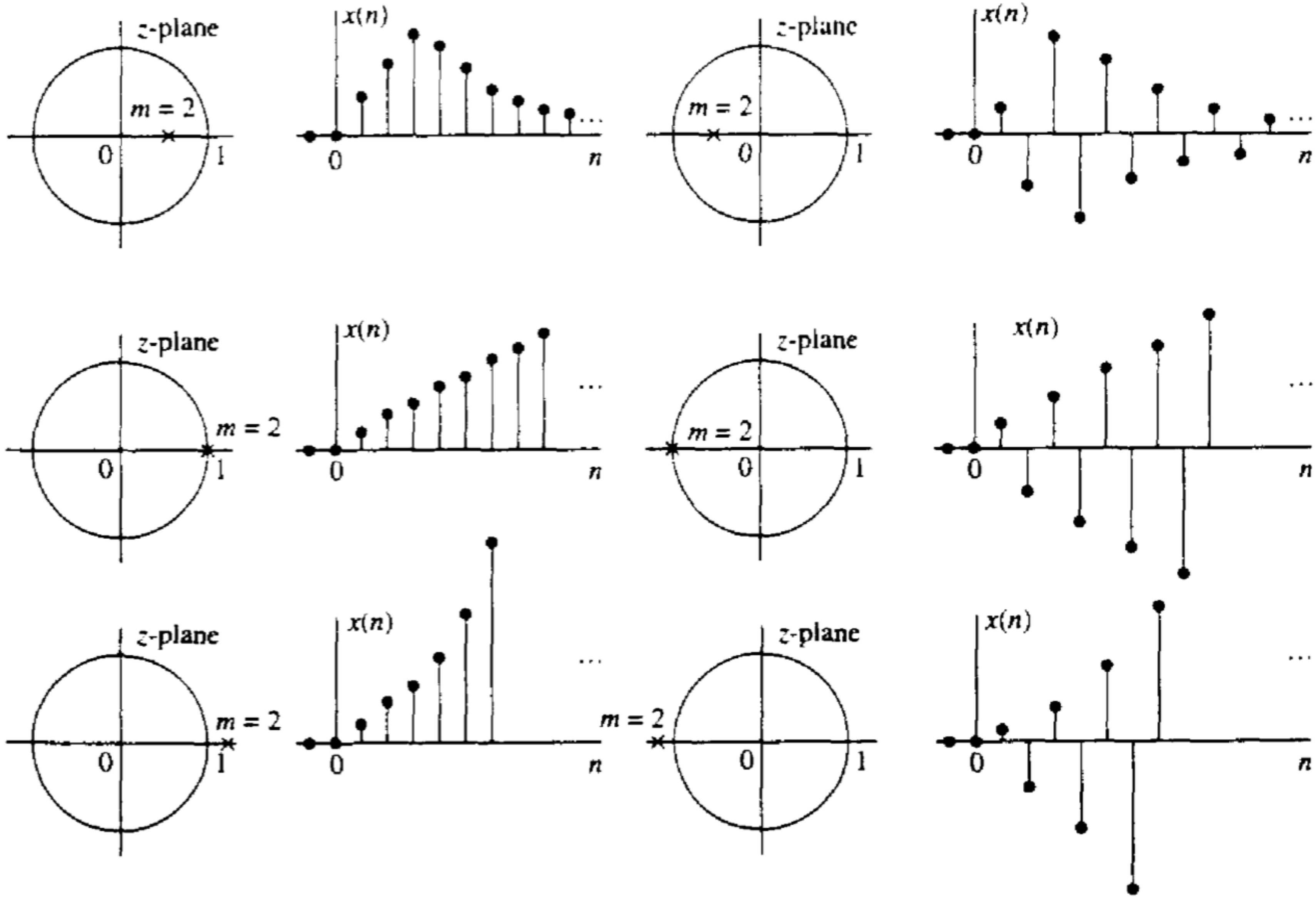
ROC: $|z| < |a|$

(convergence inside a circle)

(anticausal (left-sided) signal)

- two different signals can give the same formula for $X(z)$
- they will have different ROCs

second order poles



This figure was taken from Proakis & Manolakis "Digital Signal Processing: Principles, Algorithms and Applications,", 3rd edition, Prentice Hall.

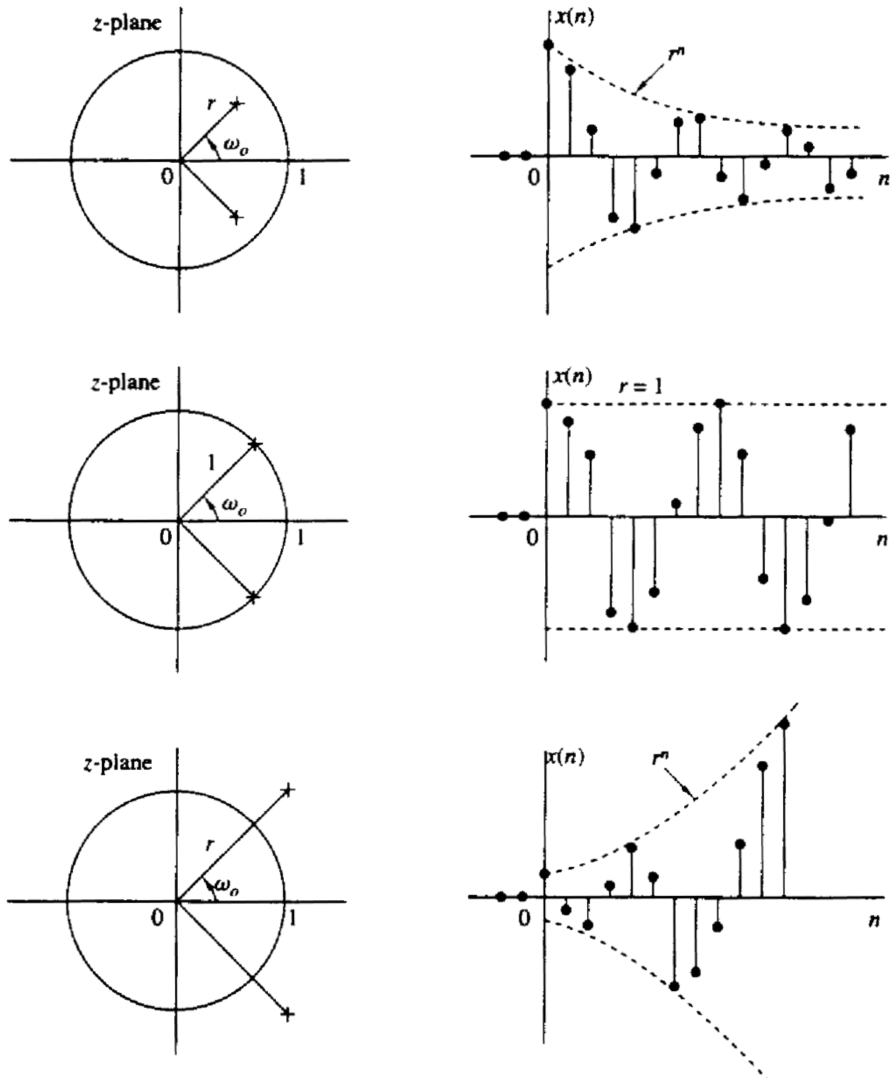
example

$$
\begin{aligned}
x[n] &= r^n \cos(\omega_0 n) u[n], \quad r > 0, \;\; 0 \le \omega_0 < 2\pi \\
&= \frac{1}{2}\left(r^n e^{j\omega_0 n} + r^n e^{-j\omega_0 n}\right) u[n] \\
&= \frac{1}{2}\left((re^{j\omega_0})^n + (re^{-j\omega_0})^n\right) u[n] \\
&= \frac{1}{2}\left(p^n + (p^*)^n\right) u[n], \quad p = re^{j\omega_0}
\end{aligned}
$$

$$
\begin{aligned}
X(z) &= \frac{1}{2}\left(\frac{1}{1 - pz^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1 - p^* z^{-1}}\right) \\
&= \frac{1 - r\cos(\omega_0) z^{-1}}{(1 - pz^{-1})(1 - p^* z^{-1})} = \frac{z(z - r\cos(\omega_0))}{(z - p)(z - p^*)}
\end{aligned}
$$

ROC : $\{z \in \mathbb{C} : |z| > |p|\} \cap \{z \in \mathbb{C} : |z| > |p|\} = \{z \in \mathbb{C} : |z| > r\}$

complex conjugate poles

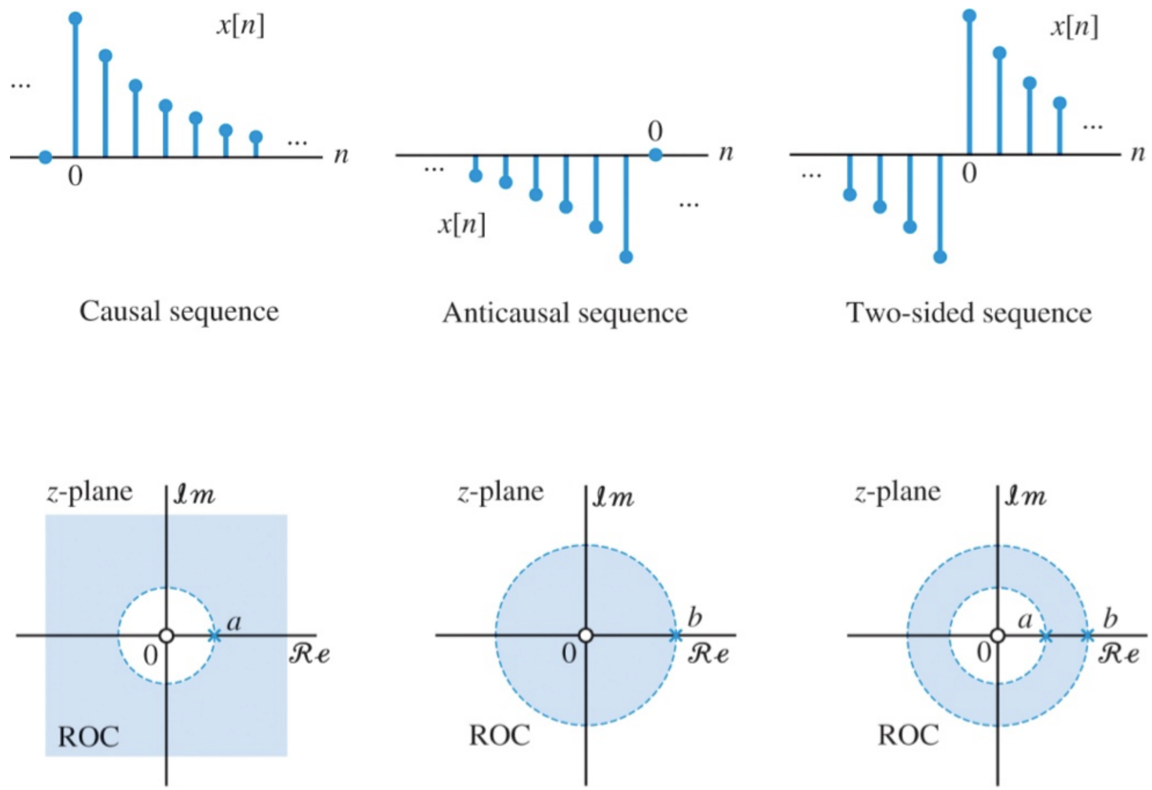


This figure was taken from Proakis & Manolakis "Digital Signal Processing: Principles, Algorithms and Applications,", 3rd edition, Prentice Hall.

two-sided exponential

$$x[n] = a^n u[n] - b^n u[-n-1] \quad \leftrightarrow \quad X(z) = \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}}$$

$$\text{ROC}: \quad |a| < |z| < |b|$$



This figure was taken from Manolakis & Ingle "Applied Digital Signal Processing,", Cambridge University Press.

- the ROC of a sum of transforms is the intersection of the ROCs

- must have $|a| < |b|$, otherwise $X(z)$ does not exist

- sequence is stable if and only if $|a| < 1 < |b|$, then unit circle in ROC

z-transform pairs

Table 3.1 Some common z-transform pairs

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

This table was taken from Manolakis & Ingle "Applied Digital Signal Processing,", Cambridge University Press.

z-transform is linear

$$x[n] \quad \leftrightarrow \quad X(z)$$

$$y[n] \quad \leftrightarrow \quad y(z)$$

$$z[n] = ax[n] + by[n] \quad \leftrightarrow \quad Z(z) = aX(z) + bY(z)$$

ROC_Z at least $\text{ROC}_X \cap \text{ROC}_Y$

example

$$X(z) = \frac{2z^2 + 0.8z - 2.2}{z^2 + 0.3z - 0.4}, \quad |z| > 0.8$$

$$Y(z) = \frac{-0.4z + 1}{z^2 + 0.5z - 0.24}, \quad |z| > 0.8$$

$$Z(z) = X(z) + Y(z) = \frac{2z^2 - 1.8z - 0.2}{z^2 - 0.8z + 0.15}, \quad |z| > 0.5$$

- Matlab: `roots([1 0.3 -0.40]) = [-0.8, 0.5]`
- Matlab: `roots([1 0.5 -0.24]) = [-0.8, 0.3]`
- Matlab: `roots([1 -0.8 0.15]) = [ 0.3, 0.5]`
- the ROC of $Z(z)$ is larger than the intersection of the ROCs of $X(z)$ and $Y(z)$ because of pole-zero cancellation when two rational functions are summed

z-transform of time shift

$$x[n] \quad \leftrightarrow \quad X(z), \quad ROC_Z$$
$$y[n] = x[n-k] \quad \leftrightarrow \quad Y(z) = z^{-k}X(z)$$

- $\text{ROC}_Y = \text{ROC}_X$ with consideration for $z = 0$ and $z = \infty$
- this is easy to derive using a single change of variables ($m = n - k$)

$$Y(z) = \sum_n y[n]z^{-n} = \sum_n x[n-k]z^{-n} = \sum_m z[m]z^{-m}z^{-k} = z^{-k}X(z)$$

example

$$
\begin{aligned}
x[n] = u[n] \quad &\leftrightarrow \quad X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots, \quad 0 < |z| \\
x[n] = u[n-1] \quad &\leftrightarrow \quad z^{-1}X(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots, \quad 0 < |z| \\
x[n] = u[n+1] \quad &\leftrightarrow \quad z^{1}X(z) = z^{1} + 1 + z^{-1} + z^{-2} + \cdots, \quad 0 < |z| < \infty
\end{aligned}
$$

$$
\begin{aligned}
x[n] = u[-n] \quad &\leftrightarrow \quad X(z) = \cdots + z^{3} + z^{2} + z + 1, \quad |z| < \infty \\
x[n] = u[-(n+1)] \quad &\leftrightarrow \quad z^{1}X(z) = \cdots + z^{4} + z^{3} + z^{2} + z, \quad |z| < \infty \\
x[n] = u[-(n-1)] \quad &\leftrightarrow \quad z^{-1}X(z) = \cdots + z^{2} + z^{1} + 1 + z^{-1}, \quad 0 < |z| < \infty
\end{aligned}
$$

- sometimes the ROC does not change
- sometimes the ROC changes by removing the point at $z = 0$ or $z = \infty$

z-transform of difference equation

- take z-transforms of both sides of the difference quation using linearity and the time-shifting property

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k]$$

$$\sum_{k=0}^{N} a[k]z^{-k}Y(z) = \sum_{k=0}^{M} b[k]z^{-k}X(z)$$

$$\left(\sum_{k=0}^{N} a[k]z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b[k]z^{-k}\right) X(z)$$

- now rearrange and define the transfer (system) function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b[k]z^{-k}}{\sum_{k=0}^{N} a[k]z^{-k}} = \frac{B(z)}{A(z)} = \sum_{n=-\infty}^{\infty} h[n]z^{-n},$$

where $h[n]$ is the impulse response

- finite poles are roots $p_k, k = 1, 2, \cdots, N$ of $A(z) = \sum_{k=0}^{N} a[k] z^{-k}$

- finite zeros are roots $z_k, k = 1, 2, \cdots, M$ of $B(z) = \sum_{k=0}^{M} b[k] z^{-k}$

- may also have poles or zeros at $z = \infty$

- counting poles and zeros at ∞, a rational function has equal numbers of poles and zeros

- the transfer function may also be written in factored form exposing the poles and zeros

$$H(z) = \frac{\displaystyle\sum_{k=0}^{M} b[k] z^{-k}}{\displaystyle\sum_{k=0}^{N} a[k] z^{-k}} = \frac{b[0]}{a[0]} \cdot \frac{\displaystyle\prod_{k=1}^{M} (1 - z_k z^{-1})}{\displaystyle\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

- Matlab `roots`, `poly` and `conv` functions

z-transform of convolution

$$
\begin{aligned}
x[n] \quad &\leftrightarrow \quad X(z) \\
y[n] \quad &\leftrightarrow \quad y(z)
\end{aligned}
$$

$$
z[n] = x[n] * y[n] \quad \leftrightarrow \quad Z(z) = X(z) \cdot Y(z)
$$

$$
\text{ROC}_Z \ \text{ at least } \ \text{ROC}_X \cap \text{ROC}_Y
$$

- this property is easy to derive using one change of variables

example

determine the output of LTI system described by $y[n] = \frac{1}{2}y[n-1] + x[n]$ driven by $x[n] = 10\cos(\pi n/4)u[n]$ and $y[-1] = 0$

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad X(z) = \frac{10(1 - \frac{1}{\sqrt{2}}z^{-1})}{1 - \sqrt{2}z^{-1} + z^{-2}}$$

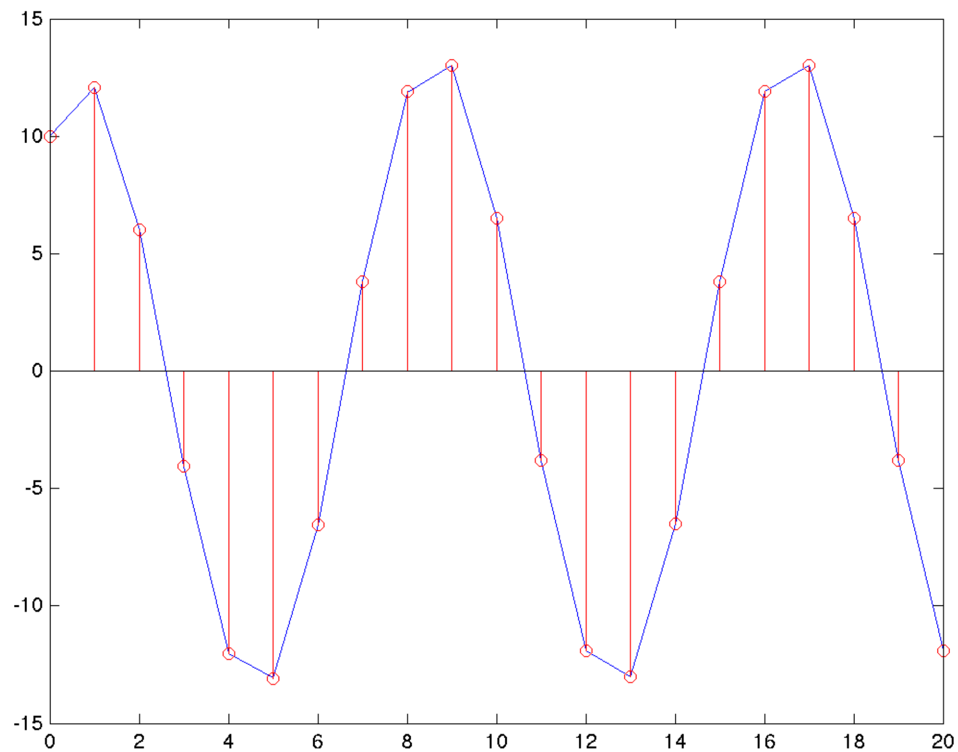
$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{10(1 - \frac{1}{\sqrt{2}}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \sqrt{2}z^{-1} + z^{-2})} \\ &= \frac{-1.9}{1 - \frac{1}{2}z^{-1}} + \frac{6.78e^{-j28.7^\circ}}{1 - e^{j\pi/4}z^{-1}} + \frac{6.78e^{j28.7^\circ}}{1 - e^{-j\pi/4}z^{-1}} \end{aligned}$$

$$\begin{aligned} y[n] &= \left(-1.9\left(\frac{1}{2}\right)^n + 6.78e^{j28.7^\circ}e^{j\pi n/4} + 6.78e^{j28.7^\circ}e^{j\pi n/4}\right)u[n] \\ &= \left(-1.9\left(\frac{1}{2}\right)^n + 13.56\cos(\pi n/4 - 28.7^\circ)\right)u[n] \end{aligned}$$

- partial fraction expansion was used (Matlab: `residuez`)

example (continued)

```
[r,p,k]=residuez(10*[1,-1/sqrt(2)],conv([1,-1/2],[1,-sqrt(2),1]));
n = [0:20];
y = -1.9074*(0.5).^n + 13.56*cos(pi*n/4-28.7*pi/180);
plot(n,y,'b'); hold on;
y = filter(1,[1,-0.5],10*cos(pi*n/4));
stem(n,y,'r'); hold off;
```



exponential modulation

$$x[n] \quad \leftrightarrow \quad X(z), \quad \text{ROC}_X$$
$$y[n] = a^n x[n] \quad \leftrightarrow \quad Y(z) = X(a^{-1}z), \quad \text{ROC}_Y = |a|\text{ROC}_X$$

- let p be a pole of $X(z)$: $X(p) \to \infty$
- then $Y(ap) = X(a^{-1}(ap)) = X(p) \to \infty$, so ap is a pole of $Y(z)$
- the ROC stretches ($|a| > 1$), shrinks ($|a| < 1$), or stays the same ($|a| = 1$)
- let $a = e^{j\omega_0}$, then pole at p gets rotated to $e^{j\omega_0}p$, but the ROC stays the same

differentiation property

$$x[n] \quad \leftrightarrow \quad X(z)$$

$$nx[n] \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$$

proof:

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} = (-z^{-1}) \sum_{n=-\infty}^{\infty} (nx[n])z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (nx[n])z^{-n}$$

- the ROC does not change

example

$$a^n u[n] \quad \leftrightarrow \quad \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$na^n u[n] \quad \leftrightarrow \quad -z\frac{d}{dz}\frac{1}{1-az^{-1}} = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

more properties

- complex conjugate

$$
\begin{aligned}
x[n] \quad &\leftrightarrow \quad X(z), \quad \text{ROC}_X \\
y[n] = x^*[n] \quad &\leftrightarrow \quad Y(z) = X^*(z^*), \quad \text{ROC}_Y = \text{ROC}_X
\end{aligned}
$$

- time-reversal

$$
\begin{aligned}
x[n] \quad &\leftrightarrow \quad X(z), \quad \text{ROC}_X \\
y[n] = x[-n] \quad &\leftrightarrow \quad Y(z) = X(z^{-1}), \quad \text{ROC}_Y = 1/\text{ROC}_X
\end{aligned}
$$

- if $\text{ROC}_X : r_1 < |z| < r_2$, then $\text{ROC}_Y : \frac{1}{r_2} < |z| < \frac{1}{r_1}$

more properties

- initial value theorem for causal $x[n]$,

$$x[0] = \lim_{z \to \infty} X(z)$$

- final value theorem for causal $x[n]$,

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

assuming the limits exist

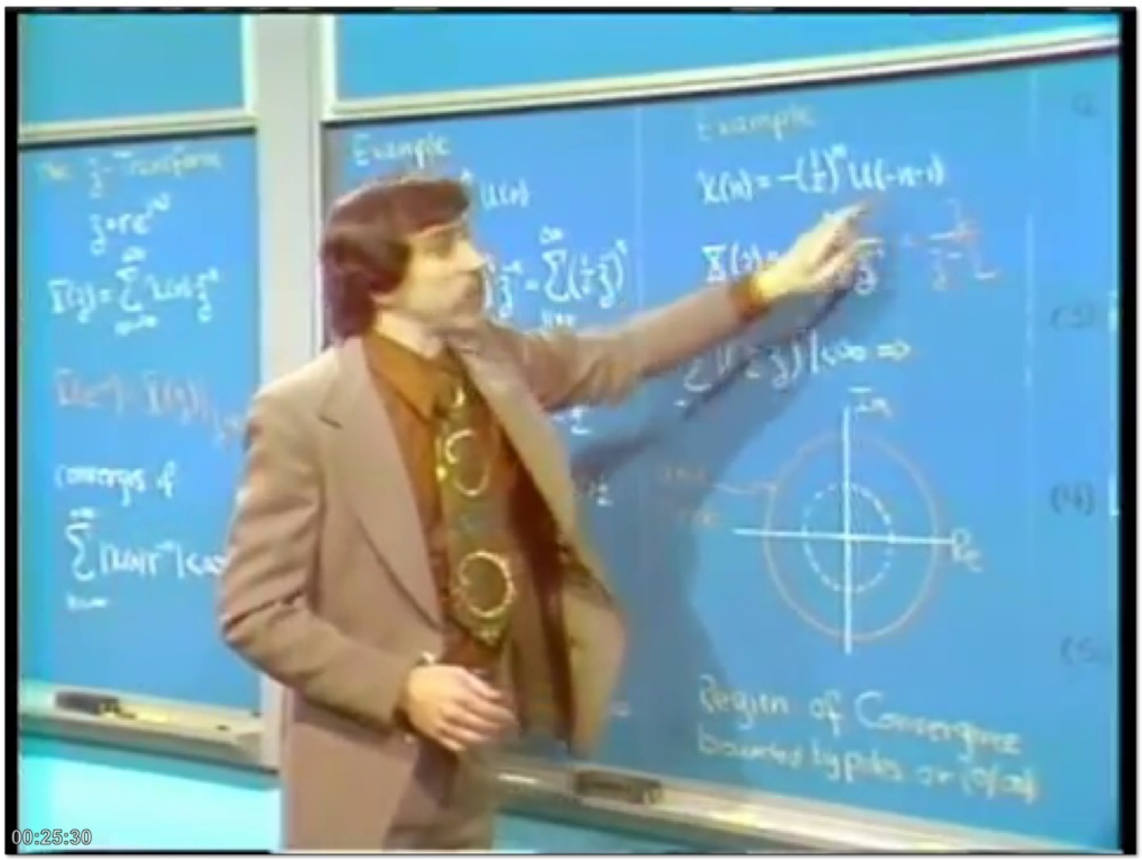
you should be able to ...

- derive the time shifting property
- derive the time-reversal property
- derive the conjugation property
- derive the convolution property
- compute z-transforms of common signals
- ... and levitate

inverse z-transform

$$x[n] = \frac{1}{2\pi j}\int_{C:CCW} X(z)z^{n-1}dz, \quad C \in ROC$$

Lecture 5 - The z-Transform.mp4
Resolution: 480x360
Filesize: 176.8 MB



observe

$$x[n] = \sum_{k=1}^{N} c_k p_k^n u[n], \quad \text{distinct } p_k$$

$$X(z) = \sum_{k=1}^{N} \frac{c_k}{1 - p_k z^{-1}}$$

$$= \frac{\sum_{k=1}^{N} c_k \prod_{m=1, m\neq k}^{N} (1 - p_m z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} = \frac{\sum_{k=0}^{N-1} b[k] z^{-k}}{\sum_{k=0}^{N} a[k] z^{-k}}, \quad a_0 = 1$$

- this is a proper rational function
- given a rational function with distinct poles, we can reverse this procedure to find $x[n]$
- use partial fraction expansion (Matlab: `residuez`)

example

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{z(z+1)}{(z-1)(z-\frac{1}{2})}$$

- poles at $p_1 = 1$ and $p_2 = \frac{1}{2}$
- zeros at $z_1 = 0$ and $z_2 = -1$
- two distinct poles gives three different possibilities for the ROC
- PFE:

```
[R,P,K]=residuez([1,1],conv([1,-1],[1,-0.5]));
[R,P,K]=residuez([1,1],poly([1,0.5]));
```

  (residue,pole) pairs = (R=4,P=1) and (R=-3, P=0.5)

$$X(z) = \frac{4}{1 - z^{-1}} + \frac{-3}{1 - \frac{1}{2}z^{-1}}$$

example continued

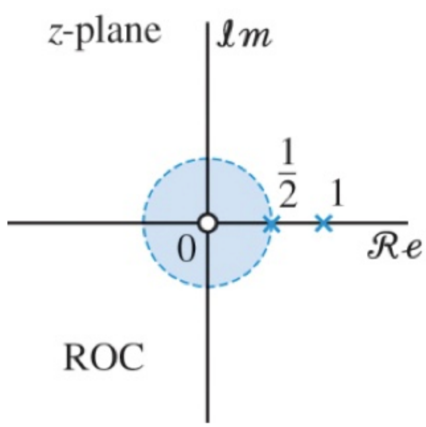
$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{4}{1 - z^{-1}} + \frac{-3}{1 - \frac{1}{2}z^{-1}}$$

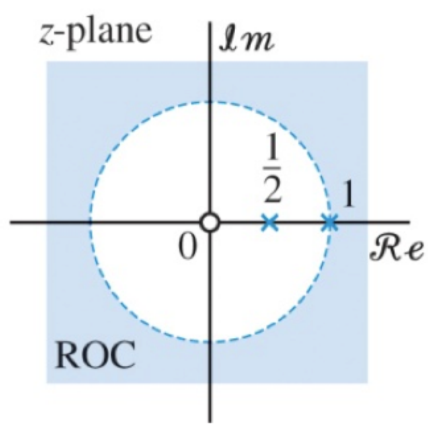
with two distinct poles $p_1 = 1, p_2 = \frac{1}{2}$, there are three possible regions of convergence

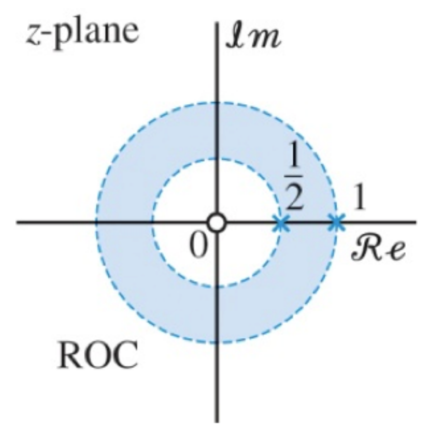
1. left-sided sequence, ROC : $|z| < \frac{1}{2}$, $\;x[n] = 3\left(\frac{1}{2}\right)^n u[-n-1] - 4u[-n-1]$

2. two-sided sequence, ROC : $\frac{1}{2} < |z| < 1$, $\;x[n] = -3\left(\frac{1}{2}\right)^n u[n] - 4u[-n-1]$

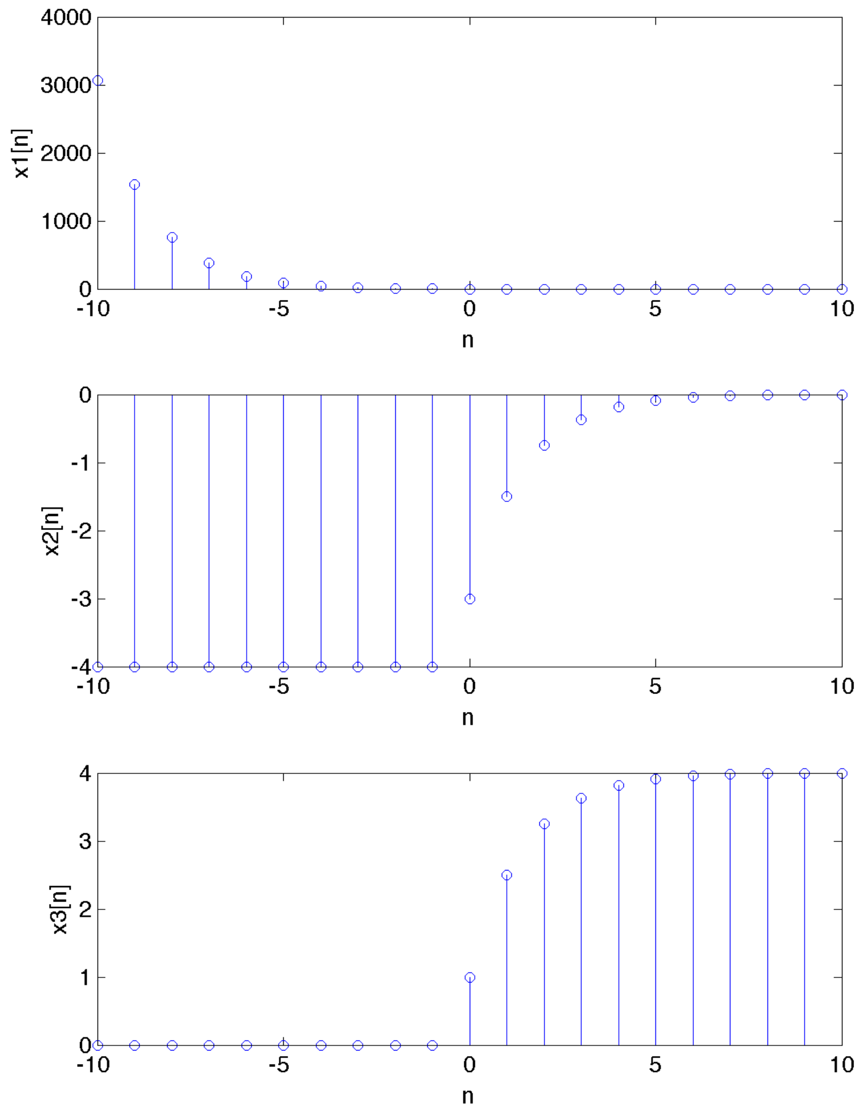
3. right-sided sequence, ROC : $1 < |z|$, $\;x[n] = -3\left(\frac{1}{2}\right)^n u[n] + 4u[n]$







example continued



none of these sequences are stable because none of the ROCs contain the unit circle

example

$x[n]$ is causal and

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^*z^{-1}}, \quad |z| > |p|$$

$$x[n] = (Ap^n + A^*p^{*n})\,u[n] = 2\Re\{Ap^n\}u[n]$$

```
[R,P,K]=residuez([1,1],[1,-1,0.5]);
```

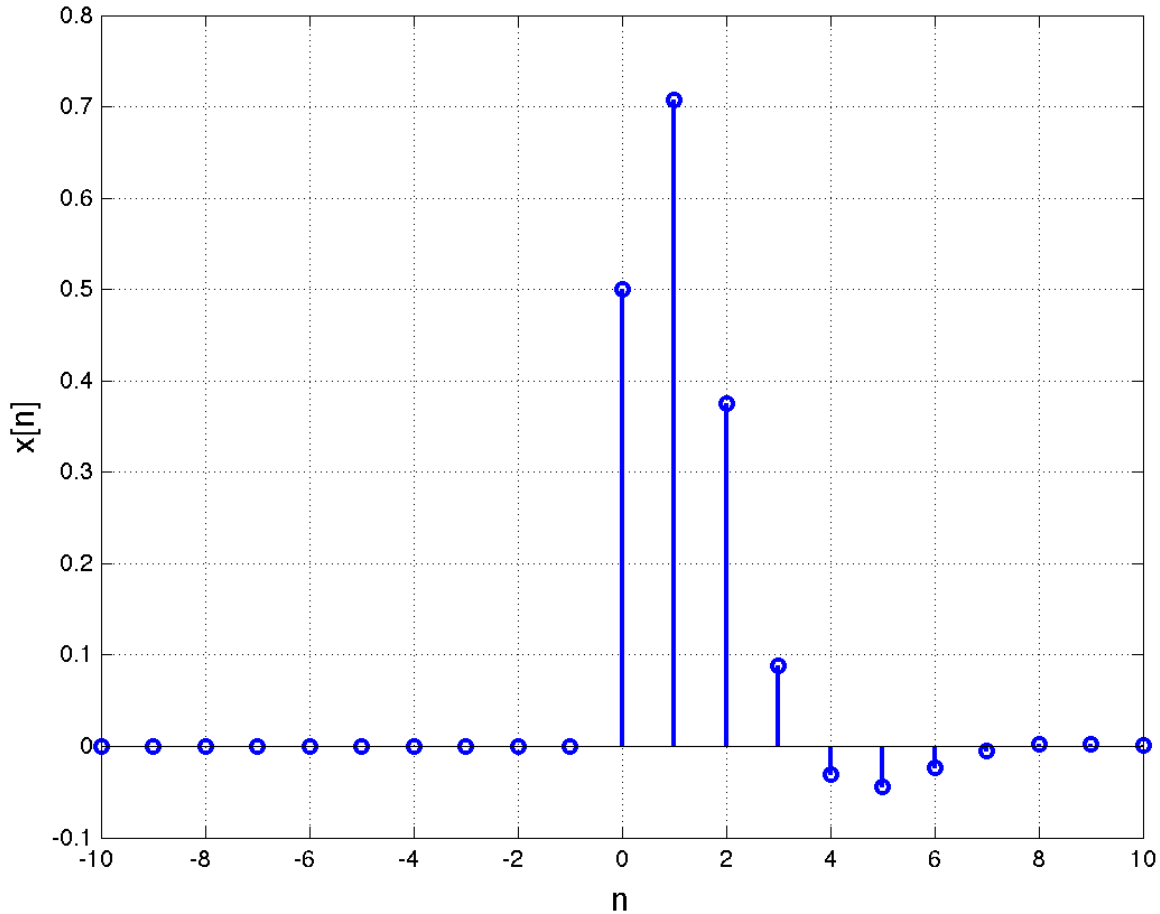
(residue,pole) pairs $= (R, P) = (\frac{1}{2} - j\frac{3}{2}, \frac{1}{2} + j\frac{1}{2})$ and $(R, P) = (\frac{1}{2} + j\frac{3}{2}, \frac{1}{2} - j\frac{1}{2})$

$$A = \frac{\sqrt{10}}{2}e^{-j71.56^\circ}, \quad p = \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}$$

$$x[n] = \frac{\sqrt{10}}{2}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n - 71.56^\circ\right) u[n]$$

is this a stable sequence? does the ROC include the unit circle?

example continued



$$x[n] = \frac{\sqrt{10}}{2}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n - 71.56^\circ\right) u[n]$$

inverse z-transform of rational functions

$$X(z) = \frac{\sum_{k=0}^{M} b[k]z^{-k}}{\sum_{k=0}^{N} a[k]z^{-k}}$$

- assume that $A(z)$ has distinct roots (distinct poles)
- PFE of $X(z)$ has the form

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

- first sum (direct term) exists when $M \geq N$
- then a causal signal $x[n]$ has the simple form

$$x[n] = \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k p_k^n u[n]$$

- but there are $N+1$ possible inverses $x[n]$, each corresponding to a different ROC

- at most one of the inverses $x[n]$ is stable

- if a pole on the unit circle, then none of the $x[n]$ are stable

LTI systems

- transfer function and impulse response of LTI system

$$H(z) = \frac{\sum_{k=0}^{M} b[k] z^{-k}}{\sum_{k=0}^{N} a[k] z^{-k}} = \sum_{n=-\infty}^{\infty} h[n] z^{-n}, \quad \mathsf{ROC}_H$$

- system is stable if ROC contains the unit circle

- system is causal if ROC lies outside the outtermost pole

- system is causal and stable if all poles lie inside the unit circle

- zeros of $H(z)$ can be anywhere

LTI systems example

is a causal realization of the system below stable?

$$H(z) = \frac{1 - z^{-2}}{1 + 0.9z^{-1} + 0.6z^{-2} + 0.05z^{-3}}$$

```
[R,P,K]=residuez([1,0,-1],[1,0.9,0.6,0.05]);
zplane([1,0,-1],[1,0.9,0.6,0.05]);
roots([1,0.9,0.6,0.05])
```

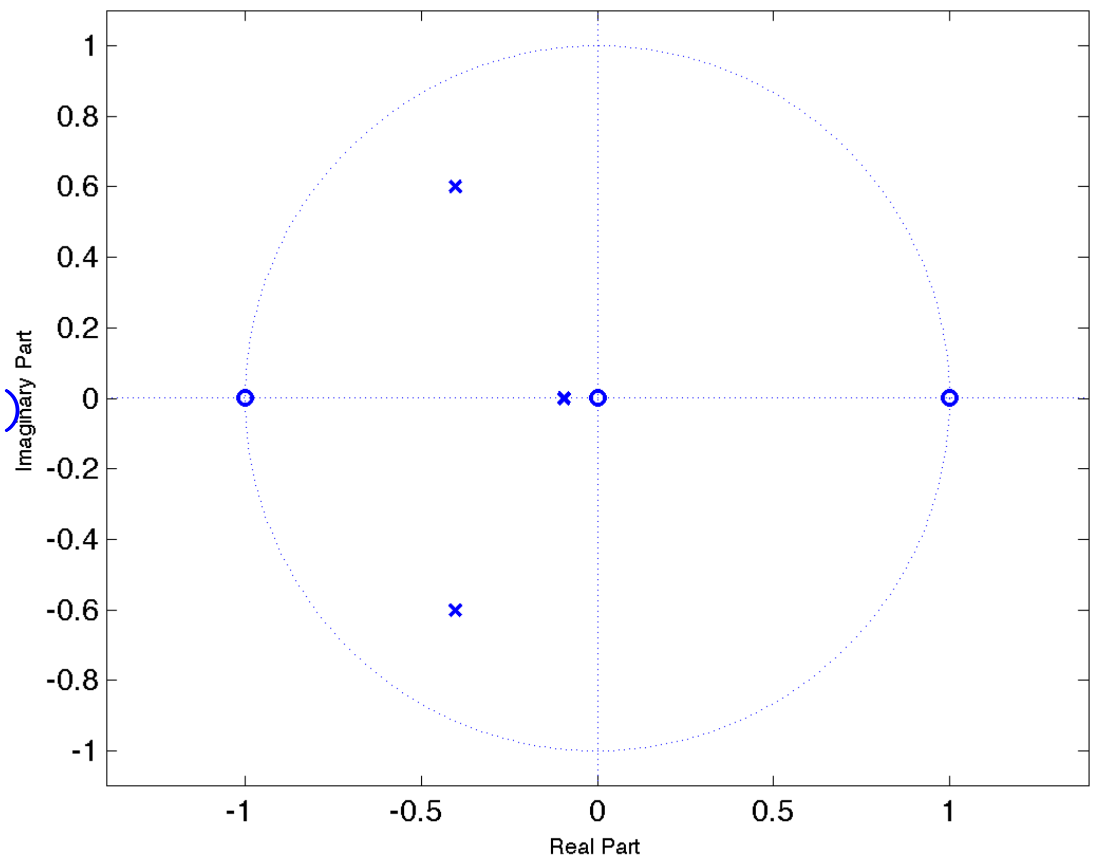
the last one produces

```
-0.4022 + 0.6011i
-0.4022 - 0.6011i
-0.0956
```

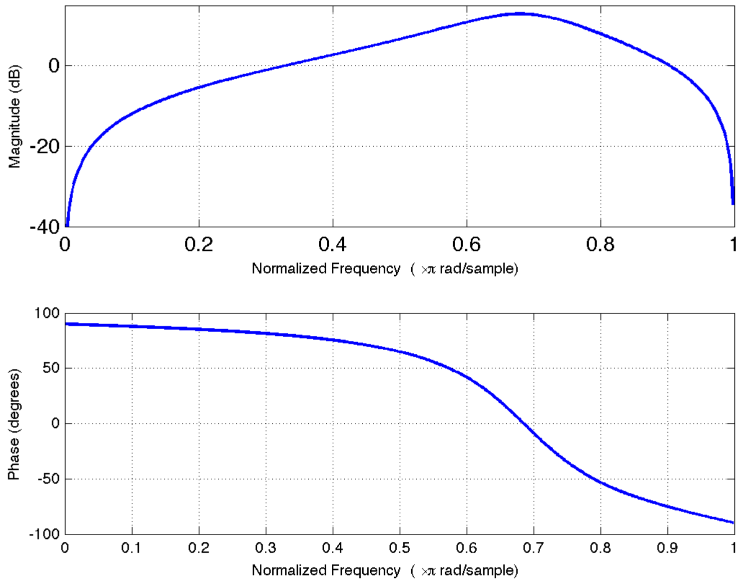
```
abs(roots([1,0.9,0.6,0.05]))
```

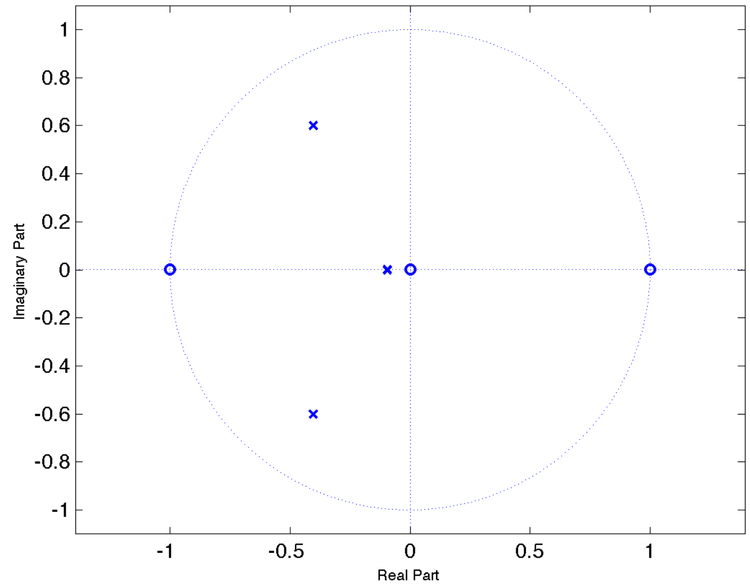
this produces

```
0.7233
0.7233
0.0956
```



LTI systems example continued






```
zplane([1,0,-1],[1,0.9,0.6,0.05]);
freqz([1,0,-1],[1,0.9,0.6,0.05]);
```

complex conjugate roots

$$H(z) = \frac{\sum_{k=0}^{M} b[k] z^{-k}}{\sum_{k=0}^{N} a[k] z^{-k}} = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{K_1} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k,0} + b_{k,1} z^{-1}}{1 + a_{k,1} z^{-1} + a_{k,2} z^{-2}}$$

$$\sum_{k=0}^{M-N} C_k \delta[n-k]$$

(direct terms)

$$\sum_{k=0}^{K_1} A_k p_k^n u[n]$$

(real distinct poles)

$$\sum_{k=0}^{K_2} 2|B_k| r_k^n \cos(\omega_k n + \theta_k) u[n]$$

(complex conjugate poles)

$$\frac{B_k}{1 - p_k z^{-1}} + \frac{B_k^*}{1 - p_k^* z^{-1}} = \frac{b_{k,0} + b_{k,1} z^{-1}}{1 + a_{k,1} z^{-1} + a_{k,2} z^{-2}}, \quad B_k = |B_k| e^{j\theta_k}, \qquad p_k = r_k e^{j\omega_k}$$

$$b_{k,0} = 2|B_k| \cos\theta_k \qquad a_{k,1} = -2r_k \cos\omega_k$$

$$b_{k,1} = -2r_k |B_k| \cos(\omega_k - \theta_k) \qquad a_{k,2} = r_k^2$$

symmetries

- if $x[n] = x^*[n]$ (i.e. $x[n]$ is real), then $X(z) = X^*(z^*)$ (prove this)

- if $X(z_0) = 0$, then $X(z_0^*) = 0$

- zeros off the real axis occur in complex conjugate pairs

- ex: $x[n] = \delta[n] + \delta[n-2] \quad \leftrightarrow \quad X(z) = 1 + z^{-2} = (1 - jz^{-1})(1 + jz^{-1})$

- `x=randn(1,5); roots(x)` returns

```
 0.5565 + 0.6921i
 0.5565 - 0.6921i
-1.3343
-0.6746
```

as expected, zeros are real or occur in complex conjugate pairs

symmetries

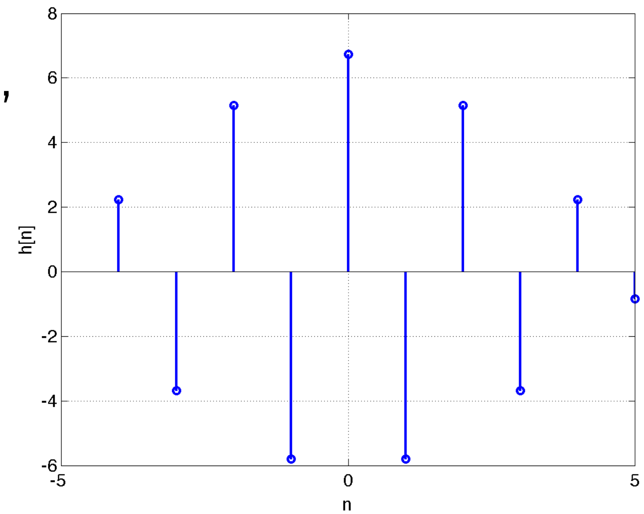
- if $x[n] = x[-n]$ (i.e. $x[n]$ is even), then $X(z) = X(z^{-1})$ (prove this)
- if $X(z_0) = 0$, then $X(z_0^{-1}) = 0$
- zeros occur in reciprocal pairs
- example:

$$x[n] = \left\{1, \frac{-35}{6}, \frac{31}{3}, \frac{-35}{6}, 1\right\}$$

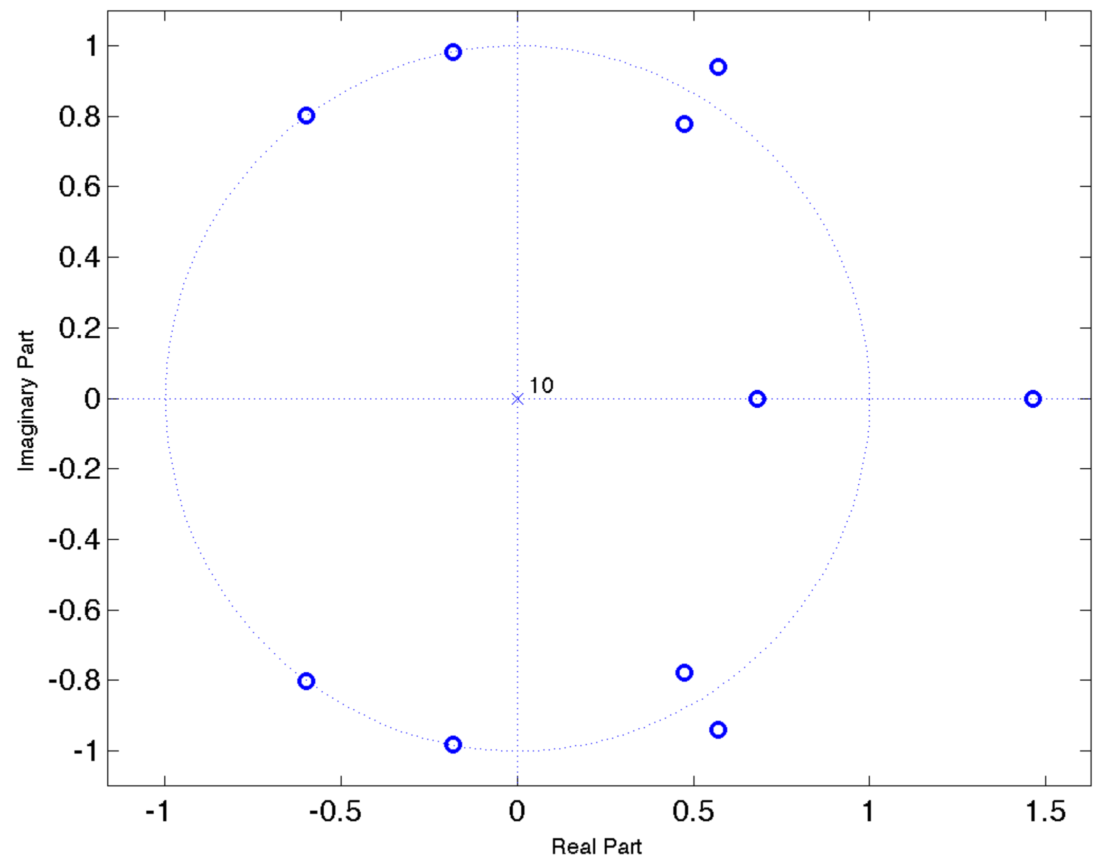
$$\begin{aligned} X(z) &= z^2 - \frac{35}{6}z^1 + \frac{31}{3} - \frac{35}{6}z^{-1} + z^{-2} \\ &= z^2\left(1 - 2z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 3z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right) \end{aligned}$$

symmetries

- if $h[n] = h[-n] = h^*[n] = h^*[-n]$ (i.e. real and even), then $H(z_0) = H(z_0^*) = H(z_0^{-1}) = H(z_0^{-*}) = 0$
- in general zeros occur in 4-tuples $(z_0, z_0^*, z_0^{-1}, z_0^{-*})$
- unit circle zeros appear in conjugate pairs (z_0, z_0^*)
- real zeros occur in reciprocal pairs (z_0, z_0^{-1})
- zeros at ± 1 can appear alone




```
h =   -0.8320
       2.2262
      -3.6809
       5.1558
      -5.7986
       6.7314
      -5.7986
       5.1558
      -3.6809
       2.2262
      -0.8320
```



one-sided z-transform

- z-transform of $x[n]u[n]$

- $x[n]$ for $n < 0$ is ignored

$$X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad \text{ROC=exterior of circle}$$

- most z-transform properties carry over to the one-sided z-transform except for the time-shifting property

$$\begin{aligned}
x[n] &\leftrightarrow X^+(z) \\
x[n-1] &\leftrightarrow x[-1] + z^{-1}X^+(z) \\
x[n-2] &\leftrightarrow x[-2] + x[-1]z^{-1} + z^{-2}X^+(z) \\
x[n-k] &\leftrightarrow \sum_{m=1}^{k} x[-m]z^{-k+m} + z^{-k}X^+(z)
\end{aligned}$$

zero-input and zero-state response of LTI system

- difference equation with non-zero initial condition

$$y[n] = ay[n-1] + bx[n], \quad n \geq 0, \qquad y[-1] \neq 0$$

- take one-sided z-transform of both sides

$$Y^+(z) = ay[-1] + z^{-1}Y^+(z) + bX^+(z)$$

- solve for $Y^+(z)$ yields

$$Y^+(z) = \underbrace{\frac{ay[-1]}{1 - az^{-1}}}_{\text{zero-input}} + \underbrace{\frac{b}{1 - az^{-1}}X^+(z)}_{\text{zero-state}}$$

zero-input and zero-state response of LTI system

consider a step input

$$x[n] = u[n] \quad \leftrightarrow \quad X^+(z) = \frac{1}{1 - z^{-1}}$$

then the output is

$$\begin{aligned} Y^+(z) &= \frac{ay[-1]}{1 - az^{-1}} + \frac{b}{(1 - az^{-1})(1 - z^{-1})} \\ &= \frac{ay[-1]}{1 - az^{-1}} + \frac{b/(1 - a)}{1 - z^{-1}} - \frac{ab/(1 - a)}{1 - az^{-1}} \\ y[n] &= \underbrace{y[-1]a^{n+1}}_{\text{zero-input}} + \underbrace{\frac{b}{1 - a}\left(1 - a^{n+1}\right)}_{\text{zero-state}}, \quad n \geq 0 \end{aligned}$$