ECE 3640 - Discrete-Time Signals and Systems
Overview of Linear Time-Invariant Systems: Part 1

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main points

- DT LTI systems model real physical systems
- behavior of real system predicted by DT LTI model
- compute the output of DT IIR LTI systems (difference equation)
- compute the output of DT FIR LTI systems (convolution)
- explain the difference between *convolution* and *filtering*
- list merits of DT LTI system relative to CT LTI system models

differential equations and difference equations

linear constant coefficient differential equation

$$\sum_{i=0}^{N} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

linear constant coefficient difference equation

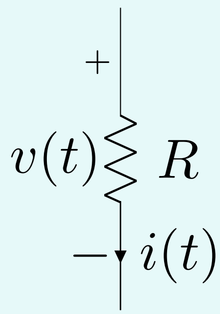
$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i] \qquad \text{(autoregressive moving-average, IIR)}$$

$$\sum_{i=0}^{N} a_i y[n-i] = x[n] \qquad \text{(autoregressive, IIR)}$$

$$y[n] = \sum_{i=0}^{M} b_i x[n-i] \qquad \text{(moving-average, FIR, convolution)}$$

circuit elements

resistor

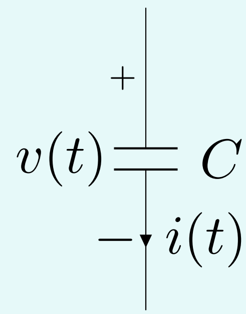


$$v(t) = Ri(t)$$

$$i(t) = \frac{v(t)}{R}$$

Impedance: R

capacitor

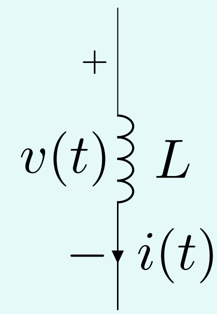


$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$i(t) = C\frac{dv(t)}{dt}$$

Impedance: $\frac{1}{sC}$

inductor



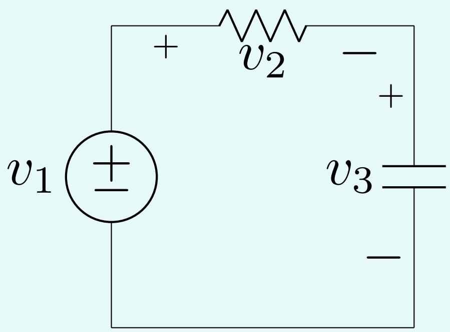
$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

Impedance: sL

circuit laws

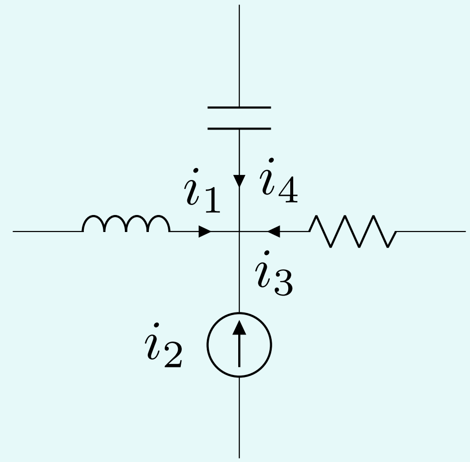
Kirchoff's voltage law (KVL)



Sum of voltage drops
around a loop is zero.

$$-v_1 + v_2 + v_3 = 0$$

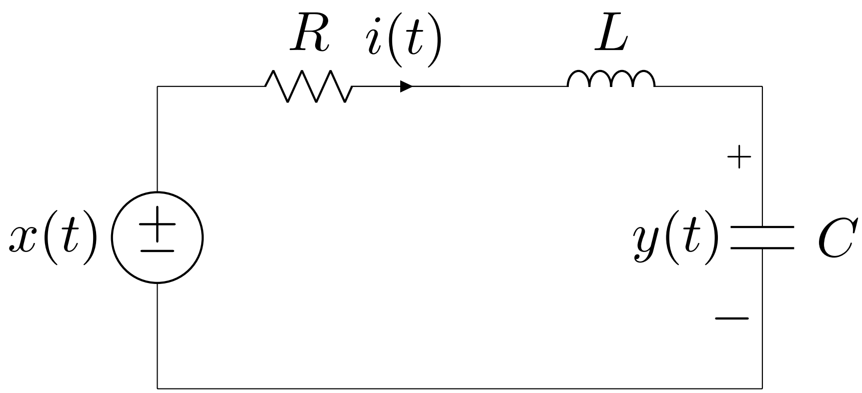
Kirchoff's current law (KCL)



Sum of currents entering
a node is zero.

$$i_1 + i_2 + i_3 + i + 4 = 0$$

series RLC circuit



Input: voltage source $x(t)$
Output: voltage across capacitor $y(t)$

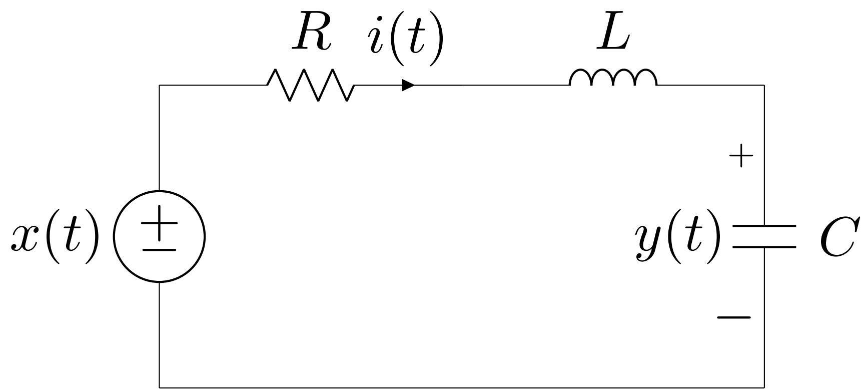
Apply KVL:

$$-x(t) + Ri(t) + L\frac{di(t)}{dt} + y(t) = 0$$

Substitute $i(t) = Cdy(t)/dt$ and rearrange

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

series RLC circuit



Input: voltage source $x(t)$
Output: voltage across capacitor $y(t)$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

linear constant coefficient differential equation (LCCDE)

- The circuit solves the differential equation.

- The differential equation models the circuit.

discretization

LCCDE solved numerically by converting to difference equations

Example: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Choose $T > 0$ and evaluate LCCDE at $t = nT$ (sample points)

$$\frac{dy(nT)}{dt} + ay(nT) = bx(nT), \quad \text{for } n = 0, 1, 2, \cdots$$

For small $T > 0$ approximate the derivative as

$$\frac{dy(t)}{dt} = \lim_{T\to 0} \frac{y(t+T) - y(t)}{T} \approx \frac{y(t+T) - y(t)}{T} \quad \text{(Euler's approx.)}$$

Then letting $x[n] = x(nT)$ and $y[n] = y(nT)$ leads to

$$\frac{y[n+1] - y[n]}{T} + ay[n] = bx[n] \quad \overset{n\to n-1}{\longrightarrow} \; y[n] + (aT - 1)y[n-1] = bTx[n-1]$$

differential and difference equations

differential equation

$$\sum_{i=0}^{N} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

- analog circuits
- physical systems
- waves

difference equation

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i]$$

- stock market
- e-mail traffic
- social networks

Physical systems solve differential/difference equations.

Differential/difference equations model physical systems.

transforms

continuous time

Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

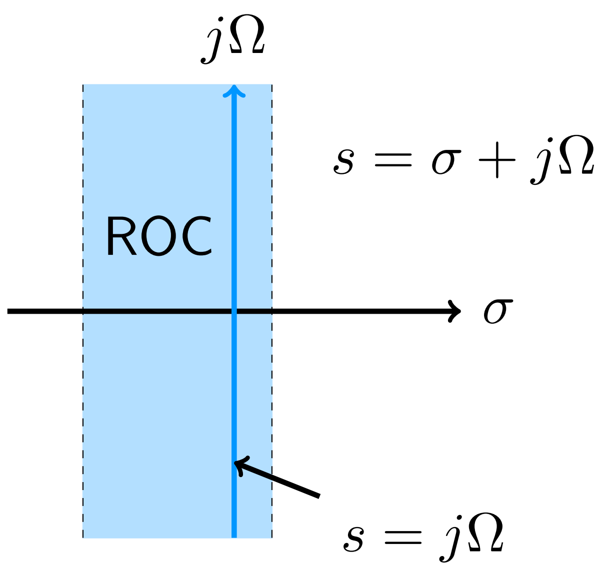
region of convergence is a vertical strip

discrete time

z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

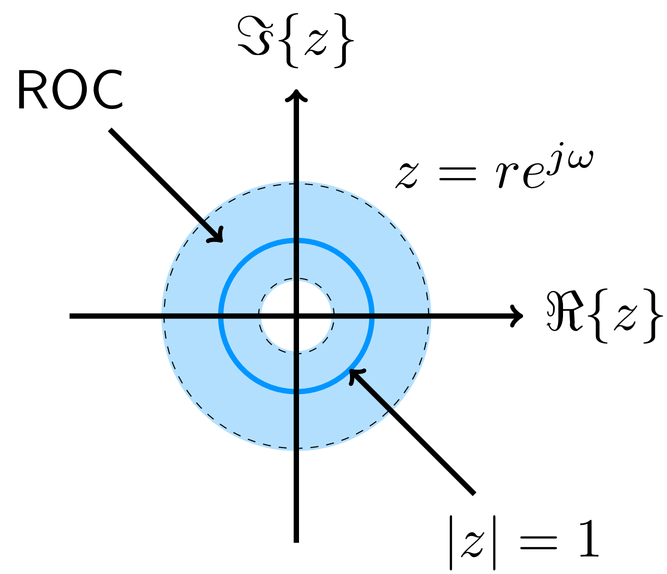
region of convergence is an annular region



$$t = nT$$

$$z = e^{sT}$$

$$\omega = \Omega T$$



system functions and solving equations

continuous time

Laplace transform derivative prop.:

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow s^k X(s)$$

Laplace transform solution:

$$\sum_{i=0}^{N} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

$$\sum_{i=0}^{N} a_i s^i Y(s) = \sum_{i=0}^{M} b_i s^i X(s)$$

$$Y(s) = H(s)X(s) = \text{P.F.E}$$

$$H(s) = \frac{\sum_{i=0}^{M} b_i s^i}{\sum_{i=0}^{N} a_i s^i}$$

discrete time

z-transform delay property:

$$x[n-k] \longleftrightarrow z^{-k} X(z)$$

z-transform solution:

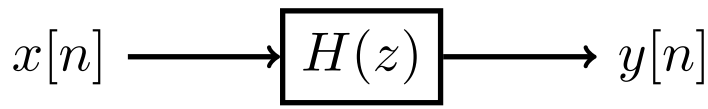
$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i]$$

$$\sum_{i=0}^{N} a_i z^{-i} Y(z) = \sum_{i=0}^{M} b_i z^{-i} X(z)$$

$$Y(z) = H(z)X(z) = \text{P.F.E}$$

$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

discrete-time linear time-invariant (LTI) systems



most general DT LTI system model is the difference equation

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i]$$

assume $a_0 = 1$ and solve for $y[n]$

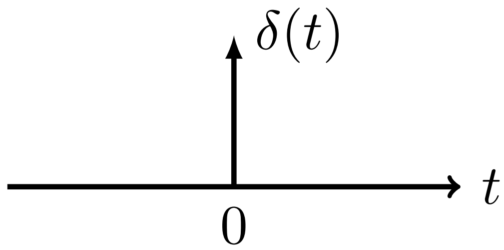
$$y[n] = \sum_{i=0}^{M} b_i x[n-i] - \sum_{i=1}^{N} a_i y[n-i]$$

compute output $y[n], n \geq 0$ recursively given initial conditions

$$x[-1], x[-2], \cdots, x[-M], y[-1], y[-2], \cdots, y[-N]$$

impulse functions and properties

Dirac delta function

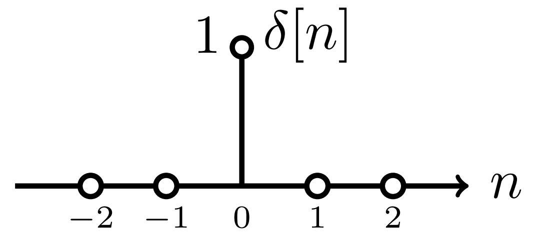


$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau = x(t)$$

Kronecker delta function



$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\sum_{k=-\infty}^{\infty} \delta[k] = 1$$

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

impulse response of LTI systems

$\delta[n] \longrightarrow \boxed{H(z)} \longrightarrow h[n]$ (impulse response)

$\delta[n-k] \longrightarrow \boxed{H(z)} \longrightarrow h[n-k]$ (time invariance)

$a\delta[n] + b\delta[n-k] \longrightarrow \boxed{H(z)} \longrightarrow ah[n] + bh[n-k]$ (linearity)

represent arbitrary input $x[n]$ as linear combination of weighted Kronecker deltas

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

use linearity and time-invariance properties to derive convolution formula

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{H(z)} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

each LTI system is characterized by its impulse response

impulse response

each LTI system is characterized by its impulse response $h[n]$

given input $x[n]$ and impulse response $h[n]$, the output $y[n]$ may be computed using the convolution sum formula

$$y[n] = h[n] * x[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

for causal LTI systems

$$h[n] = 0 \quad \text{for } n < 0$$

for (bounded-input, bounded-output) stable LTI systems

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

impulse response

frequency response function

$$H(f) = \sum_{n=\infty}^{\infty} h[n]e^{-j2\pi fn} \qquad \text{(discrete-time Fourier transform, DTFT)}$$

transfer function

$$H(z) = \sum_{n=\infty}^{\infty} h[n]z^{-n} \qquad (z\text{-transform})$$

length classes of LTI systems

when the support of $h[n]$ is $[0, \infty)$:

- the system has "infinite impulse response" (IIR)
- compute the output recursively via the difference equation

$$y[n] = \sum_{i=0}^{M} b_i x[n-i] - \sum_{i=1}^{N} a_i y[n-i] \qquad \text{(IIR)}$$

when the support of $h[n]$ is $[0, M]$ where $M < \infty$:

- the system is "finite impulse response" (FIR)
- compute output via the difference equation (convolution)

$$y[n] = \sum_{i=0}^{M} h[i]x[n-i] = \sum_{i=n-M}^{n} x[i]h[n-i] \qquad \text{(FIR)}$$

where $h[i] = b_i$

terminology: convolution vs. filtering

convolution ("block" input, "block" output)

- given $x[n], n = 0, 1, \cdots, L$ and $h[n], n = 0, 1, \cdots, M$, compute $y[n], n = 0, 1, \cdots, L + M$
- given $x[n], n = 0, 1, 2, \cdots$ and $h[n], n = 0, 1, 2, \cdots$, derive analytical expression for $y[n], n = 0, 1, 2, \cdots$

filtering

- apply filter to semi-infinite data stream
- samples arrive one at a time or a few at a time
- compute filter output one sample at a time or a few at a time

computer assignments

FIR case

- linear shift data buffer (not efficient)
- circular data buffer (efficient)
- DFT-based convolution (very efficient)
- inner product (sequential) vs. linear combination (parallel)
- multi-rate (up and down sampling)
- special structures (e.g. CIC)
- applications: audio filtering, filter banks, global positioning, pulse compression (radar), etc.
- FIR filter design
- (need to be expert in filter design, implementation, and use)

computer assignments

IIR case

- realizations
- applications: notch, bandpass, etc.
- IIR filter design
- (need to be expert in filter design, implementation, and use)

signal analysis

- use FFT to implement DFT efficiently
- analyze signals
- analyze systems
- (need to be expert in the use of the FFT)

DSP

continuous time

- CT-LTI system is a circuit
- circuit is difficult to modify
- components are inexpensive
- need lots of components to experiment with changes
- less flexible
- no sampling needed

discrete time

- DT-LTI system is a C program
- software is easy to modify
- development kits can be expensive
- only need one part (DSP + Codec) to experiment
- flexibility to implement arbitrary functions
- ADC and DAC converters necessary