ECE 3640 - Discrete-Time Signals and Systems

Discrete-Time LTI Systems: Impulse Response

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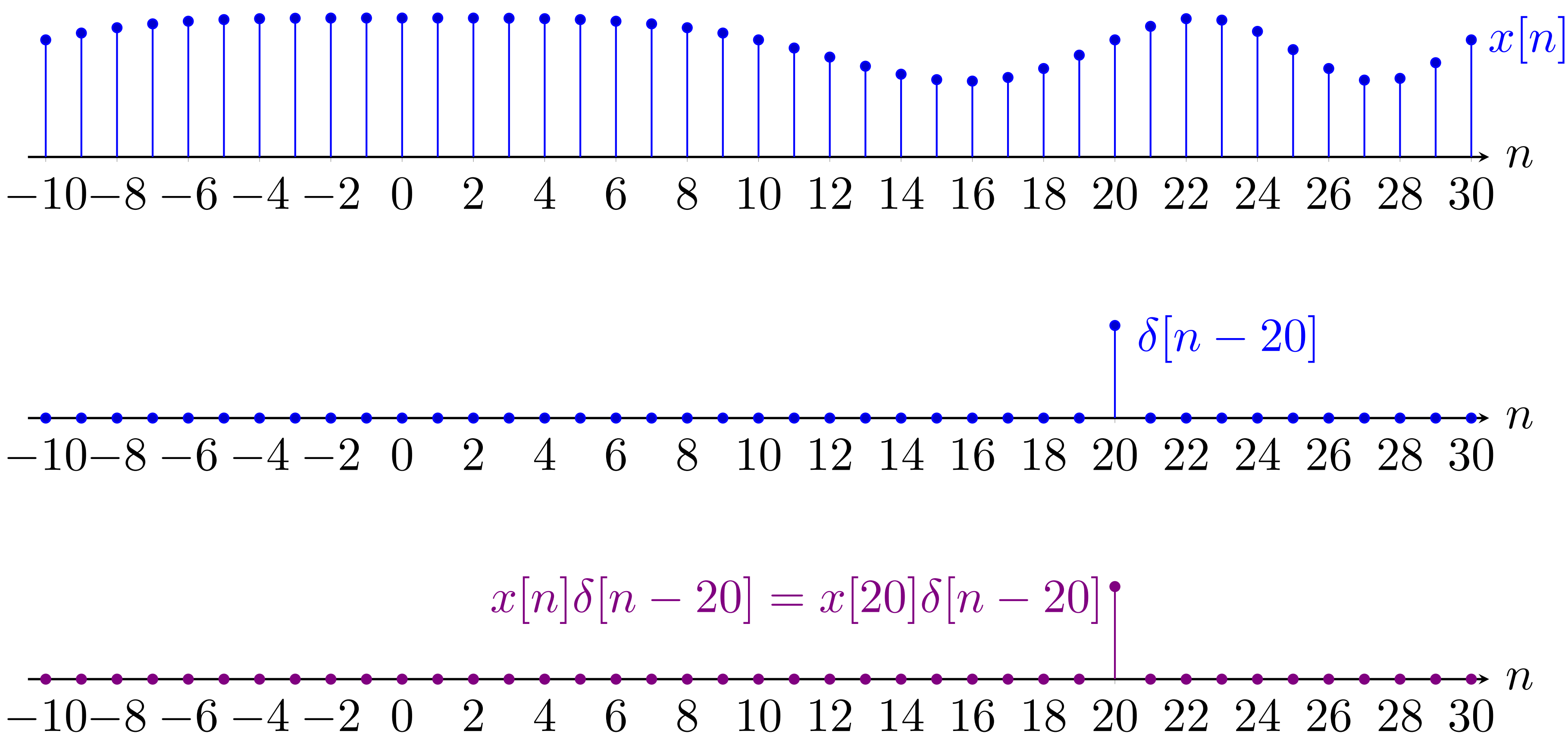
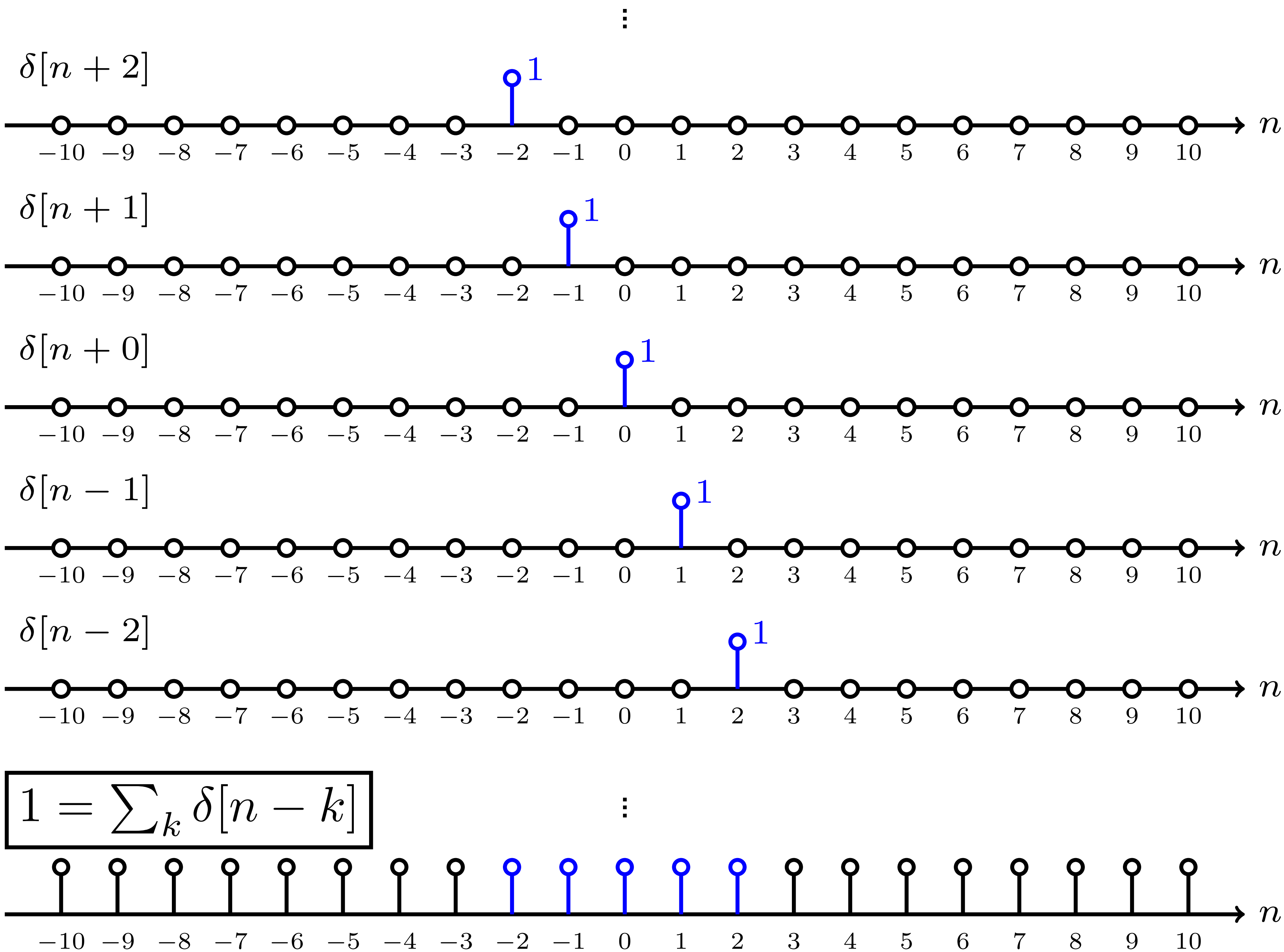
sifting property of Kronecker delta

$$x[n]\delta[n-k] = x[k]\delta[n-k] \quad \text{for all} \quad n$$



representing a constant signal



$$1 = \sum_k \delta[n-k]$$

representation using Kronecker delta

Note that:

$$1 = \sum_k \delta[n-k]$$

Now multiply signal $x[n]$ by 1 and then sift:

$$x[n] = x[n] \cdot 1 = x[n] \underbrace{\left(\sum_k \delta[n-k] \right)}_{1} = \sum_k x[n]\delta[n-k] = \sum_k x[k]\delta[n-k]$$

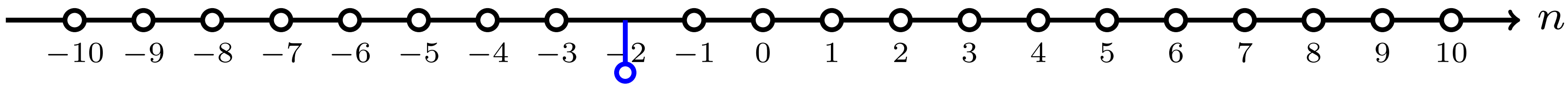
We have:

$$\boxed{x[n] = \sum_k x[k]\delta[n-k] = x[n] * \delta[n]}$$

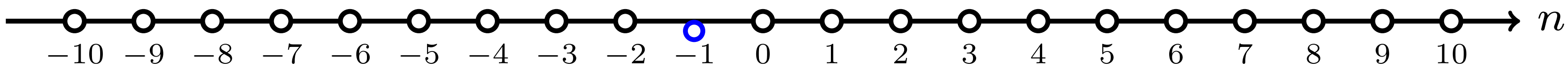
representation using Kronecker delta

⋮

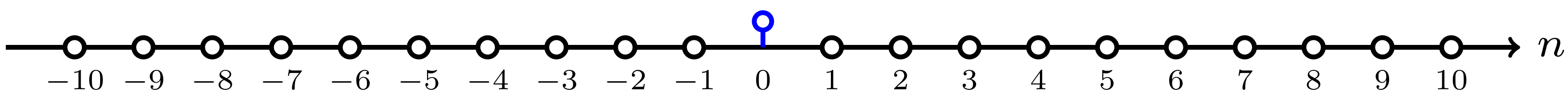
$x[n]\delta[n+2] = x[-2]\delta[n+2]$



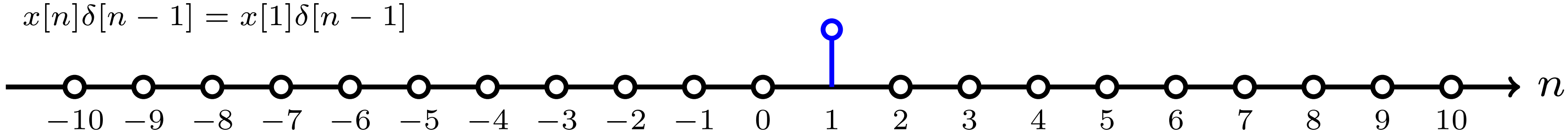
$x[n]\delta[n+1] = x[-1]\delta[n+1]$



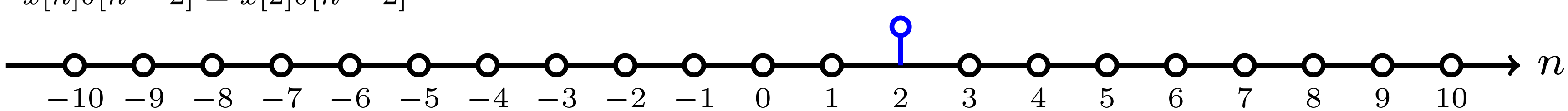
$x[n]\delta[n] = x[0]\delta[n]$



$x[n]\delta[n-1] = x[1]\delta[n-1]$

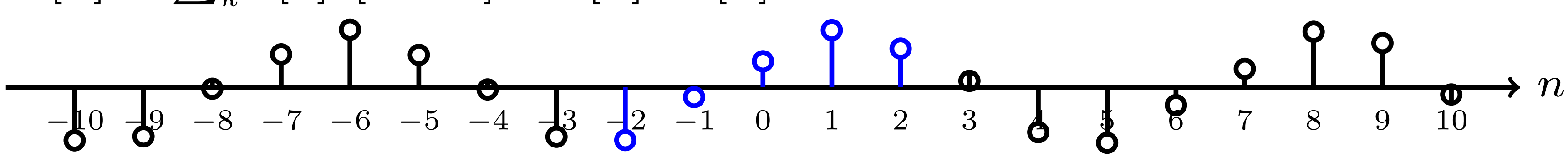


$x[n]\delta[n-2] = x[2]\delta[n-2]$

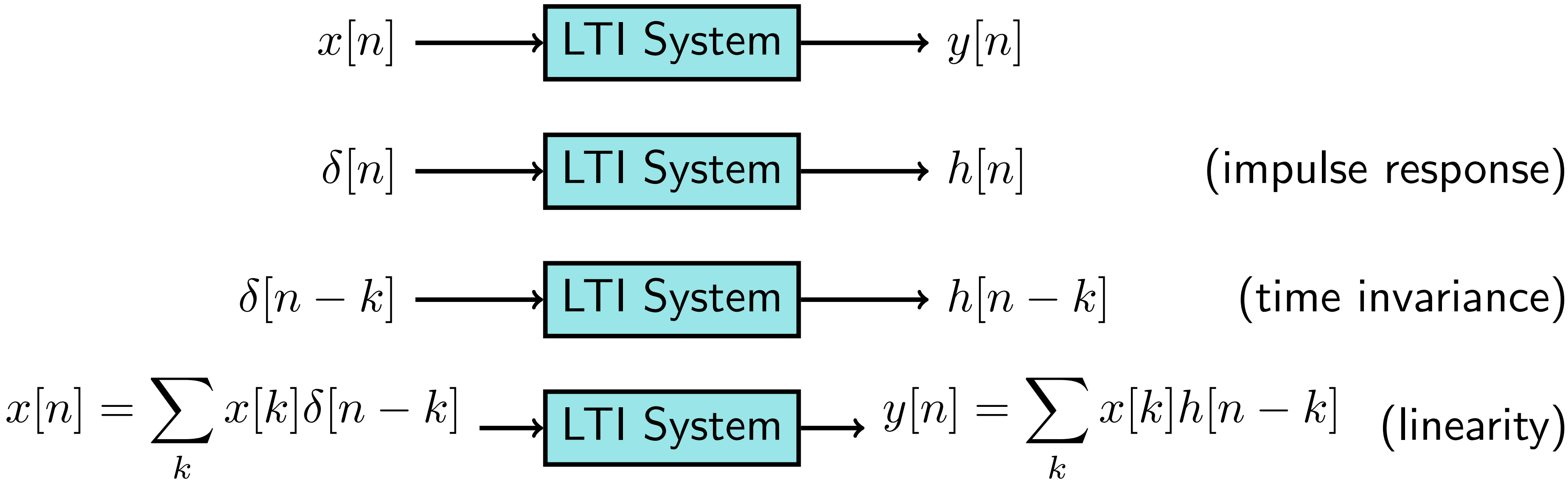


⋮

$$x[n] = \sum_k x[k]\delta[n-k] = x[n] * \delta[n]$$



impulse response of LTI systems

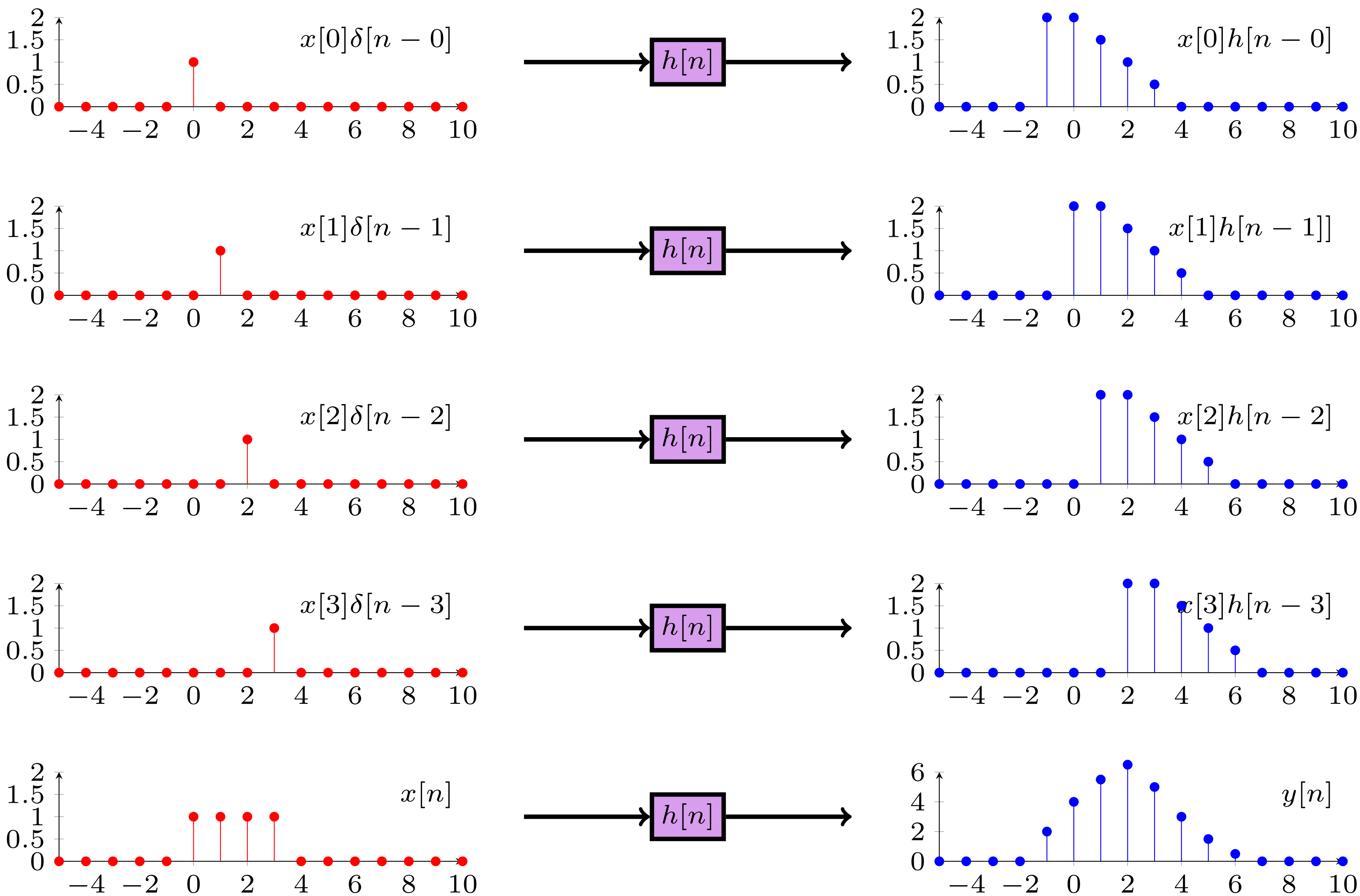


$$x[n] = x[n] * \delta[n] = \sum_k x[k]\delta[n-k] \qquad \text{(input)}$$

$$y[n] = x[n] * h[n] = \sum_k x[k]h[n-k] \qquad \text{(output)}$$

the output is the superposition of overlapping delayed replicas of the impulse response

convolution: illustration of LTI system properties



$$x[n] = \sum_k x[k]\delta[n-k]$$

$$y[n] = \sum_k x[k]h[n-k]$$

two forms of convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Change variables using $m = n - k$ or $k = n - m$

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m]$$

impulse response tells us about LTI system stability

Suppose input is bounded: $|x[n]| \le A$ for all n. Is the output bounded?

$$\begin{aligned} |y[n]| &= \left| \sum_k x[k]h[n-k] \right| \\ &\le \sum_k |x[k]h[n-k]| \qquad \text{(triangle inequality)} \\ &= \sum_k |x[k]| \cdot |h[n-k]| \\ &\le A \sum_k |h[n-k]| < \infty \end{aligned}$$

provided $h[n]$ is absolutely summable, i.e.

$$\sum_k |h[k]| < \infty$$

LTI system is stable if impulse response is absolutely summable.

impulse response tells us about LTI system causality

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$= \cdots + x[n-2]h[2] + x[n-1]h[1] \qquad \text{(past inputs)}$$

$$+ x[n]h[0] \qquad \text{(present input)}$$

$$+ x[n+1]h[-1] + x[n+2]h[-2] + \cdots \qquad \text{(future inputs)}$$

For causality $y[n]$ should not depend on future inputs, i.e. $x[n+1], x[n+2], \cdots$.
Set $h[n] = 0$ for $n < 0$.

LTI system is causal if impulse response is causal.

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{n} h[n-k]x[k] \qquad \text{(causal systems)}$$

examples

$h[n] = 2^n u[n]$	causal, unstable
$h[n] = u[n]$	causal,unstable
$h[n] = a^n u[n], \quad \lvert a \rvert < 1$	causal, stable
$h[n] = \begin{cases} \frac{1}{2N+1}, & -N \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$	non-causal, stable
$h[n] = \delta[n-d], \quad d > 0$	causal, stable
$h[n] = \delta[n+1] - \delta[n]$	non-causal, stable
$h[n] = \delta[n] - \delta[n-1]$	causal, stable

- LTI FIR systems are always stable
- Non-causal LTI FIR systems can be delayed to be causal
- Have to check stability in IIR LTI systems
- FIR/IIR distinction is new for discrete-time systems
- In continuous-time, all impulse responses are IIR

example

$$h[n] = a^{|n|}, \quad |a| < 1 = \begin{cases} a^{-n}, & n < 0 \\ 1, & n = 0 \\ a^n, & n > 0 \end{cases}$$

Is it stable?

$$\sum_{n=-\infty}^{\infty} |h[n]| = \frac{1+|a|}{1-|a|} < \infty \qquad \text{yes, it is stable}$$

Is it causal? No.

How could we make this system causal?

1. Truncate tails when $|h[n]| < \epsilon$. This makes it FIR.

2. Delay to make it causal.

This gives causal, stable LTI system that approximates the original system.

summary

- Convolution formula for LTI systems is derived using:
  1. sifting property of Kronecker delta function
  2. representation of the input using Kronecker delta functions
  3. time invariance
  4. linearity

- output of LTI system computed for any input by convolution with impulse response

- impulse response $h[n]$ of LTI system tells us about
  1. causality: $h[n] = 0$ for $n < 0$
  2. stability: $\sum_n |h[n]| < \infty$