ECE 3640 - Discrete-Time Signals and Systems
Complex Exponential Signals: $e^{j2\pi Ft}, e^{j2\pi fn}$

Jake Gunther

UtahStateUniversity

Department of Electrical & Computer Engineering

abbreviations & notation

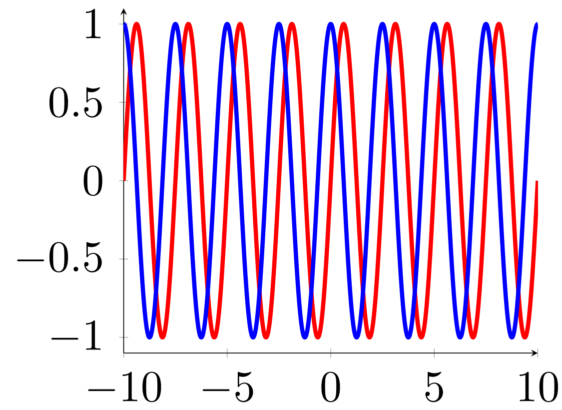
- complex exponential = CE, $e^{j2\pi Ft}, e^{j2\pi fn}$
- continuous-time = CT, $t \in \mathbb{R}$
- discrete-time = DT, $n \in \mathbb{Z}$
- set of real numbers = $\mathbb{R}$
- set of integers = $\mathbb{Z}$
- set of rational numbers = $\mathbb{Q}$
- set of complex numbers = $\mathbb{C}$

types of CT complex exponential signals

> The same types are defined for DT CE signals.

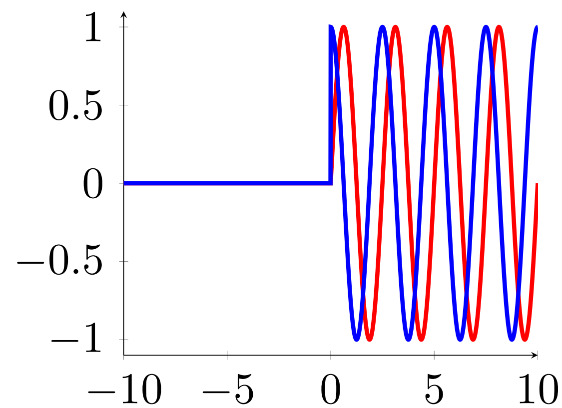
everlasting CT CE:

$$e^{j2\pi Ft}, -\infty < t < \infty$$



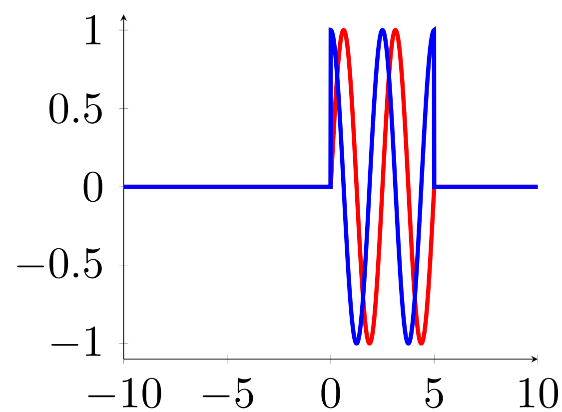
causal CT CE:

$$e^{j2\pi Ft}u(t) = \begin{cases} e^{j2\pi Ft}, & 0 \le t < \infty \\ 0, & -\infty < t < 0 \end{cases}$$



finite (windowed) CT CE:

$$e^{j2\pi Ft}[u(t) - u(t-W)] = \begin{cases} e^{j2\pi Ft}, & 0 \le t < W \\ 0, & t < 0 \text{ or } t \ge W \end{cases}$$



CT & DT CE signals

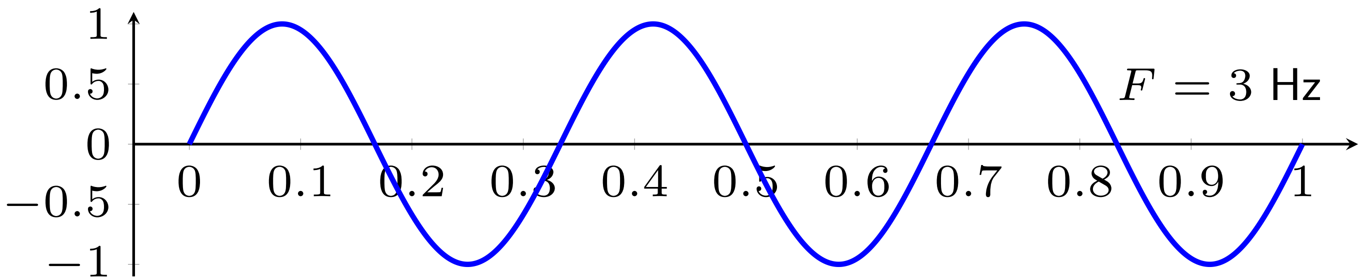
$$\left.\begin{array}{c} e^{j2\pi Ft} \\ \\ -\infty < t < \infty \\ t \in \mathbb{R} \\ \\ -\infty < F < \infty \\ F \in \mathbb{R} \end{array}\right] \xrightarrow[\substack{t = nT = n/F_s \\ f = FT = F/F_s}]{\text{sampling}} \left[\begin{array}{c} e^{j2\pi fn} \\ \\ -\infty < n < \infty \\ n \in \mathbb{Z} \\ \\ -\infty < f < \infty \\ f \in \mathbb{R} \end{array}\right.$$

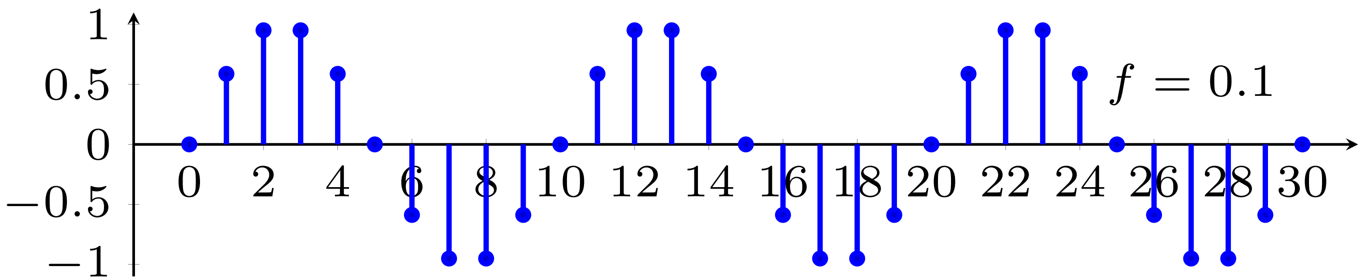
- $f = FT = F/F_s$ is often referred to as "normalized" frequency
- T is the sample period or sample interval [seconds/sample]
- $F_s = 1/T$ is the sample rate or sample frequency [samples/second]

units

$$F\left[\text{Hz} = \frac{\text{cycles}}{\text{second}}\right]$$

$$f = F\left[\text{Hz} = \frac{\text{cycles}}{\text{second}}\right] \cdot T\left[\frac{\text{seconds}}{\text{sample}}\right] = (FT)\left[\frac{\text{cycles}}{\text{sample}}\right]$$





angular and cyclic frequency

$$\Omega \left[\frac{\text{radians}}{\text{second}}\right] = 2\pi \left[\frac{\text{radians}}{\text{cycle}}\right] F \left[\frac{\text{cycles}}{\text{cycle}}\right]$$

$$\omega \left[\frac{\text{radians}}{\text{sample}}\right] = 2\pi \left[\frac{\text{radians}}{\text{cycle}}\right] f \left[\frac{\text{cycles}}{\text{sample}}\right]$$

(I am going to use cyclic frequency on the following pages.)

periodic signals

- continuous-time:

$$x(t + kT) = x(t), \text{for all } t \in \mathbb{R} \text{ and } k \in \mathbb{Z}$$

  Periodic with period $T \in \mathbb{R}$ [seconds].

- discrete-time

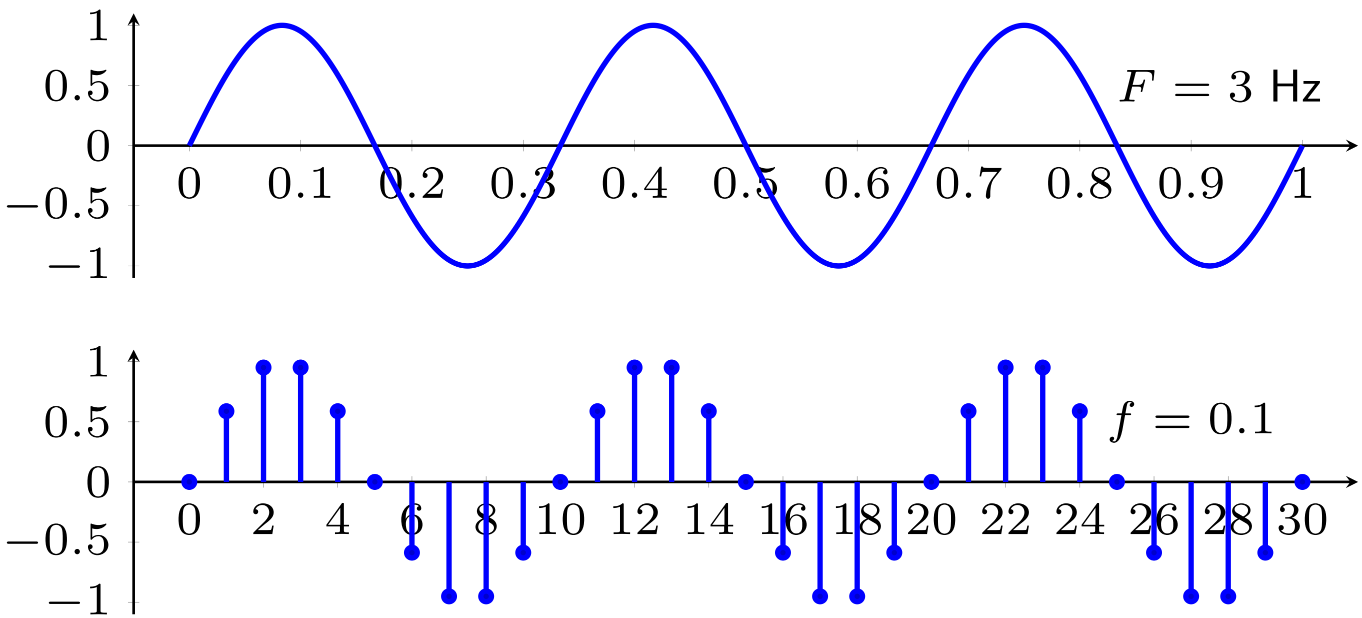
$$x[n + kN] = x[n], \text{for all } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}$$

  Periodic with period $N \in \mathbb{Z}$ [samples].

The period is always the smallest number T or N satisfying these definitions.

periodic signals

What are frequency and period?



periodic CT CE signals

$$\begin{aligned}
x(t) &= e^{j2\pi Ft} \\
x(t+T) &= x(t), \text{for all } t \in \mathbb{R} \\
e^{j2\pi Ft}e^{j2\pi FT} &= e^{j2\pi Ft}, \text{for all } t \in \mathbb{R} \\
e^{j2\pi FT} &= 1 \\
FT &= 1 \quad \text{(time } T \text{ when CE completes one full cycle)} \\
F &= \frac{m}{T} \\
F &= \frac{1}{T}
\end{aligned}$$

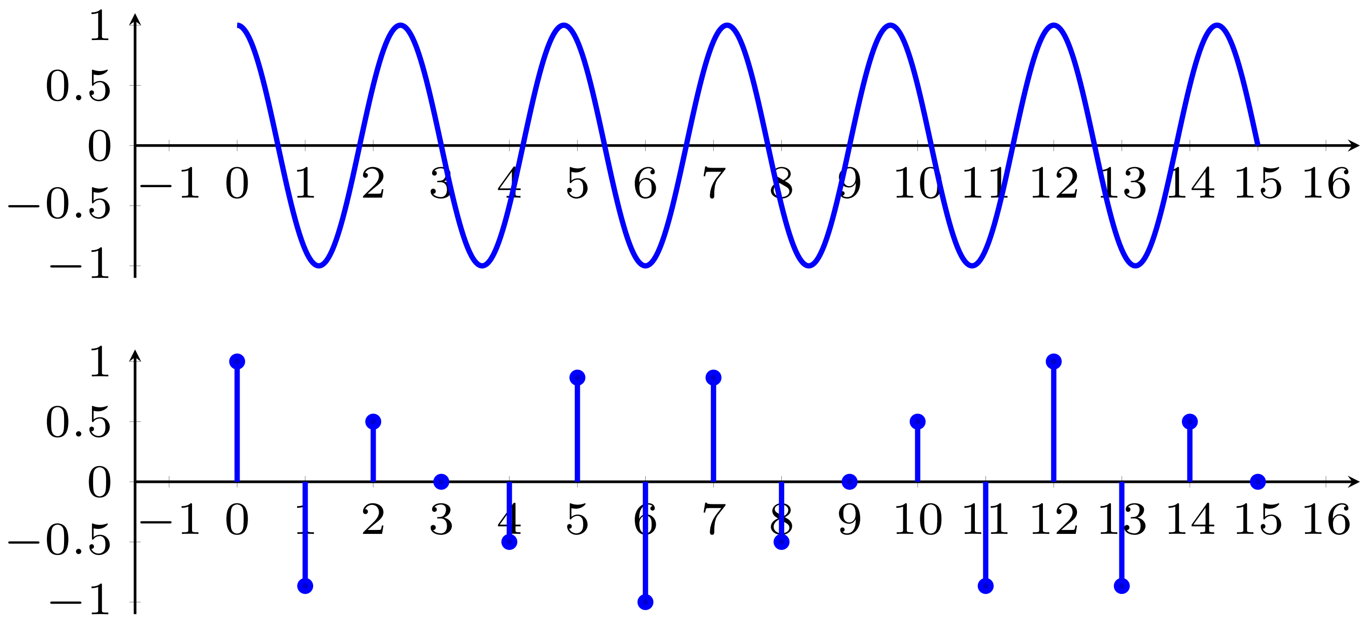
- $e^{j2\pi Ft}$ is periodic for all frequencies
- the period is $T = 1/F$

periodic DT CE signals

$$
\begin{aligned}
x[n] &= e^{j2\pi fn} \\
x[n+N] &= x[n], \text{for all } n \in \mathbb{Z} \\
e^{j2\pi fn}e^{j2\pi fN} &= e^{j2\pi fn}, \text{for all } n \in \mathbb{Z} \\
e^{j2\pi fN} &= 1 \\
fN &= m \in \mathbb{Z} \quad \text{(sample } N \text{ when CE completes } m \text{ cycles)} \\
f &= \frac{m}{N} \\
f &= \frac{p}{Q} \quad \text{(reduce fraction)}
\end{aligned}
$$

- $e^{j2\pi fn}$ is periodic only for rational frequencies
- the period is Q

periodic DT CE signals



- Discrete sequence has period $N = 12$
- Envelope completes $m = 5$ cycles
- Frequency $f = \frac{5}{12}$

CT CE are unique

$$\text{suppose} \quad e^{j2\pi F_1 t} = e^{j2\pi F_2 t} \quad \text{for all } t$$

$$e^{j2\pi(F_1 - F_2)t} = 1 \quad \text{for all } t$$

$$(F_1 - F_2)t \in \mathbb{Z} \quad \text{for all } t$$

$$F_1 - F_2 = 0$$

$$F_1 = F_2$$

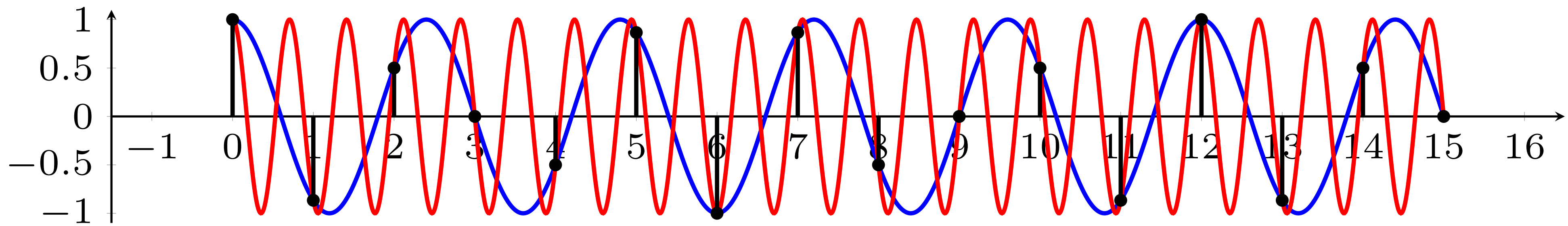
- Same signal $\Rightarrow$ same frequency
- Different frequency $\Rightarrow$ different signals

DT CE are not unique

$$\text{suppose} \quad e^{j2\pi f_1 n} = e^{j2\pi f_2 n} \quad \text{for all } n$$
$$e^{j2\pi(f_1 - f_2)n} = 1 \quad \text{for all } n$$
$$(f_1 - f_2)n \in \mathbb{Z} \quad \text{for all } n$$
$$f_1 - f_2 \in \mathbb{Z}$$
$$f_1 = f_2 + k, \quad k \in \mathbb{Z}$$

- Same signal $\Rightarrow$ frequencies separated by an integer
- Frequencies separated by non-integer $\Rightarrow$ different signals

periodic DT CE signals

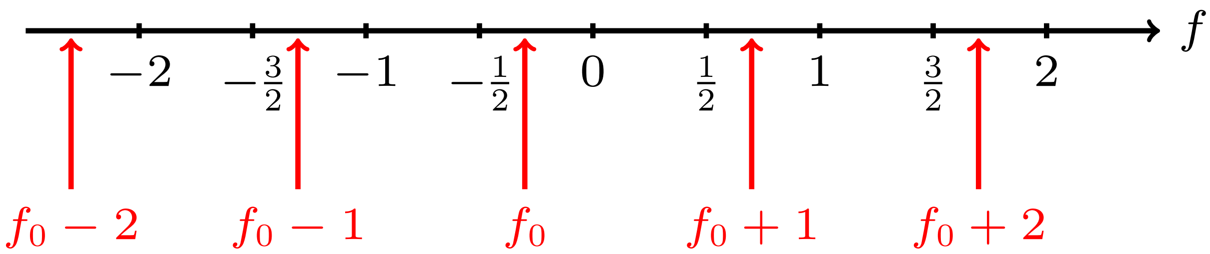


What is plotted?

- $\cos\left(2\pi\left(\frac{5}{12}\right)t\right)$
- $\cos\left(2\pi\left(\frac{5}{12}+1\right)t\right)=\cos\left(2\pi\left(\frac{17}{12}\right)t\right)$
- $\cos\left(2\pi\left(\frac{5}{12}\right)n\right)=\cos\left(2\pi\left(\frac{17}{12}\right)n\right)$ $\leftarrow$ different frequencies, same samples!

CT CE signals with different frequencies are different.
DT CE signals with frequencies related by $f_2=f_1+k$ are the same.
This is called aliasing.
Aliasing caused by sinusoids passing through same points at sample times.

aliasing

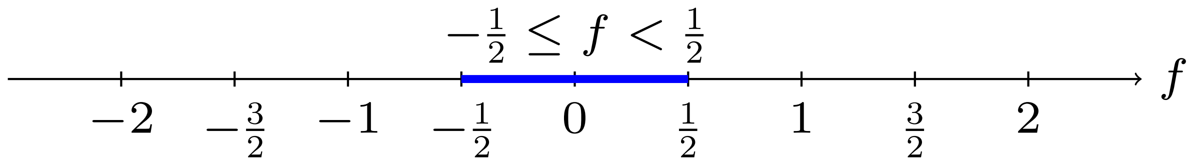


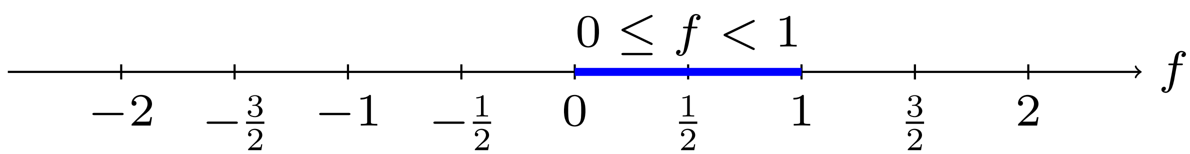
These frequencies all give the same complex exponential sequence:

$$e^{j2\pi(f_0-2)n} = e^{j2\pi(f_0-1)n} = e^{j2\pi f_0 n} = e^{j2\pi(f_0+1)n} = e^{j2\pi(f_0+2)n}, \quad \text{for all } n.$$

- these signals are all aliases of one another
- high frequencies alias to low frequencies
- different CT frequencies give distinct CT CE waveforms
- different DT frequencies have aliases yielding identical DT CE sequences

fundamental interval of unique DT frequencies





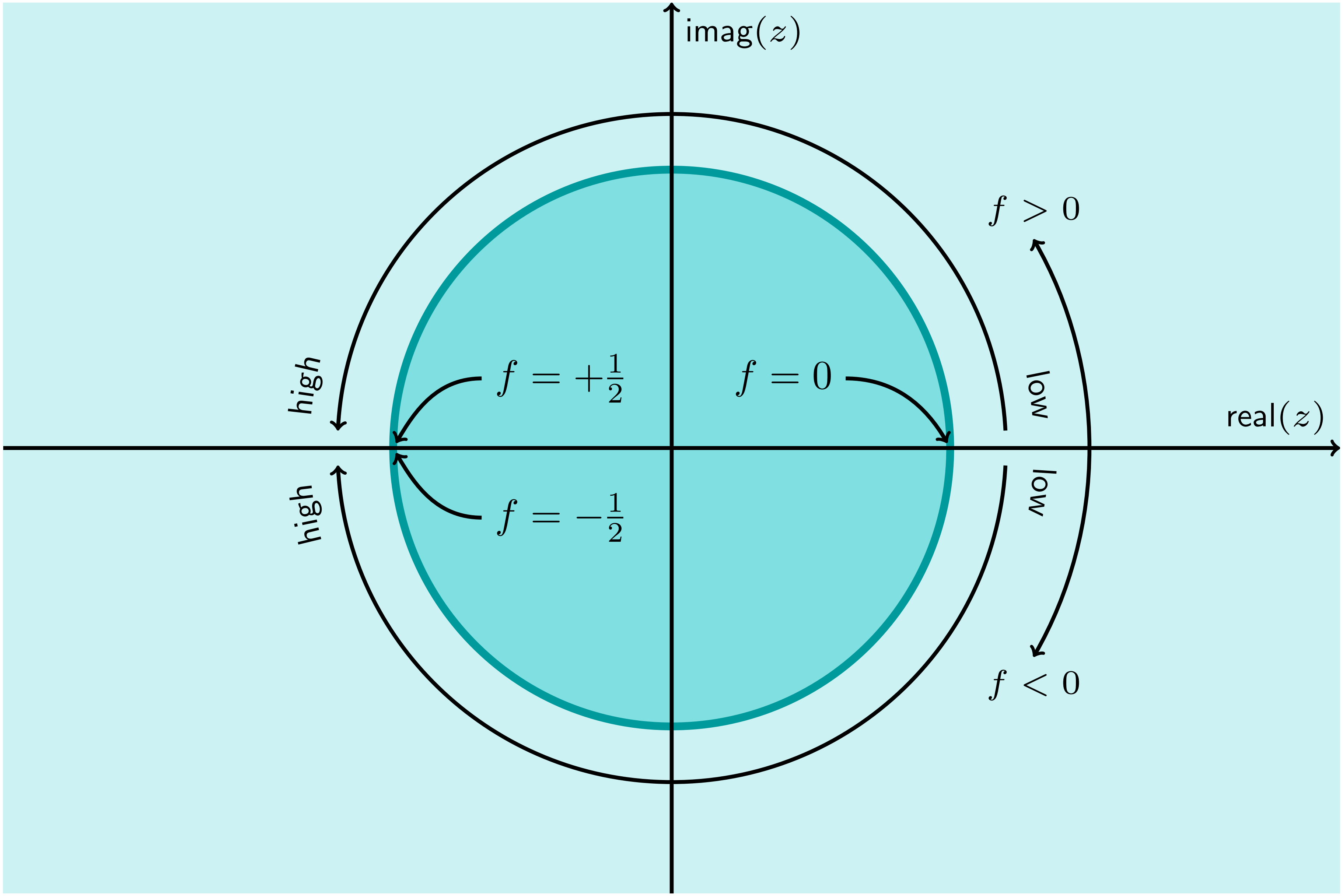
- Fundamental interval has width of 1 [cycle/sample].

- Fundamental interval can be placed anywhere, but these two choices are common.

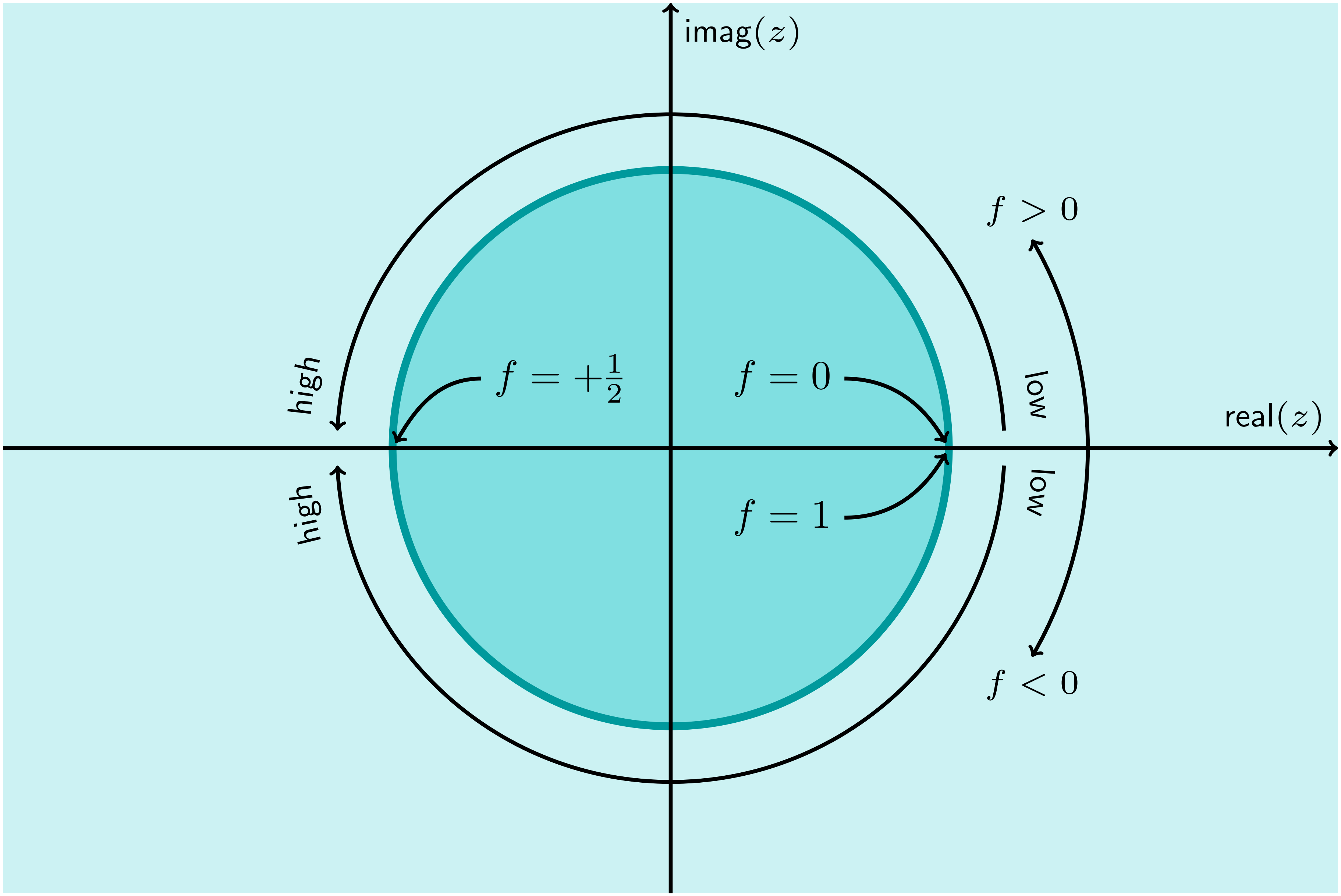
- Fundamental interval contains unique frequencies.

- Frequencies outside fundamental interval have aliases in fundamental interval.

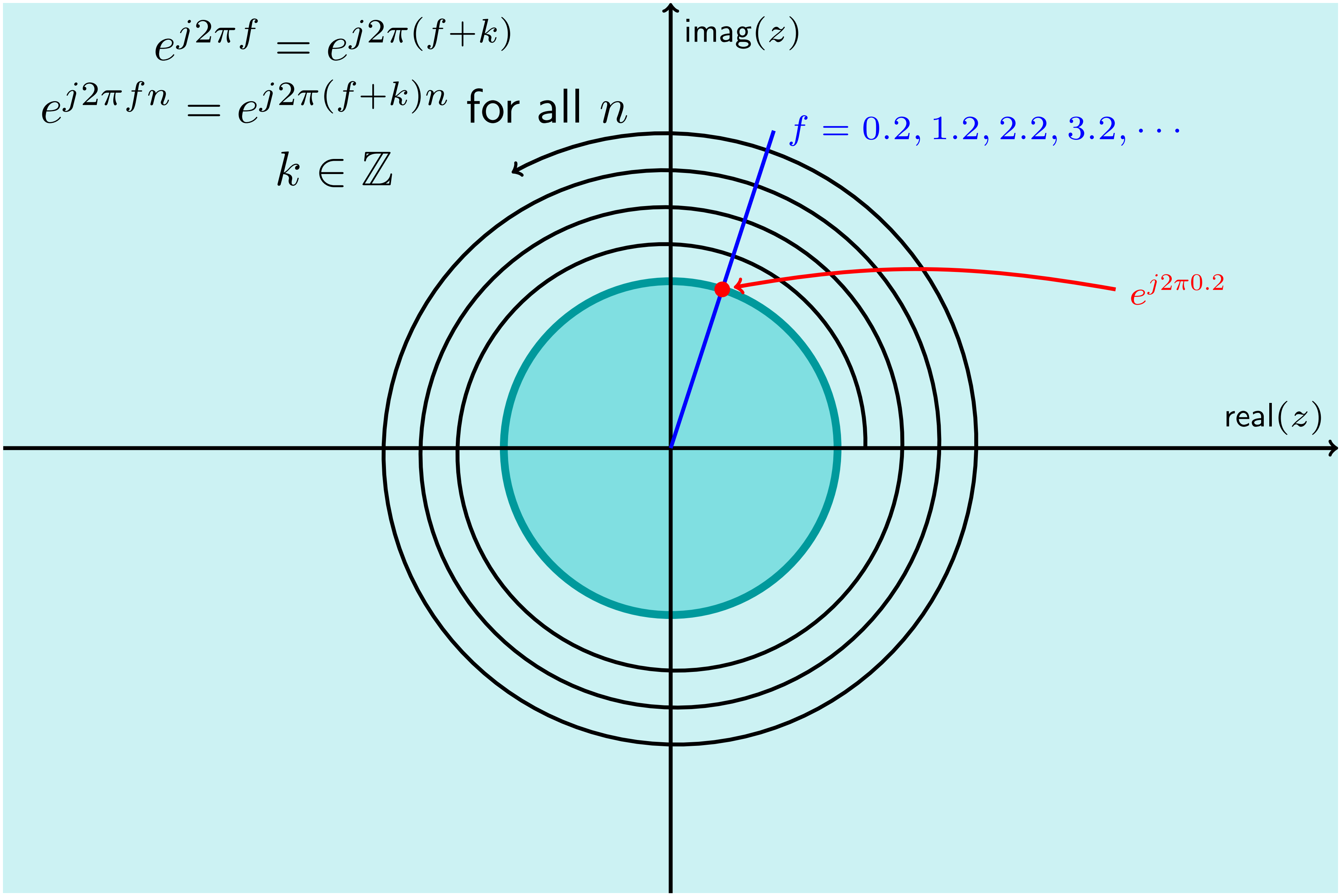
fundamental frequency interval $-\frac{1}{2} \leq f < \frac{1}{2}$



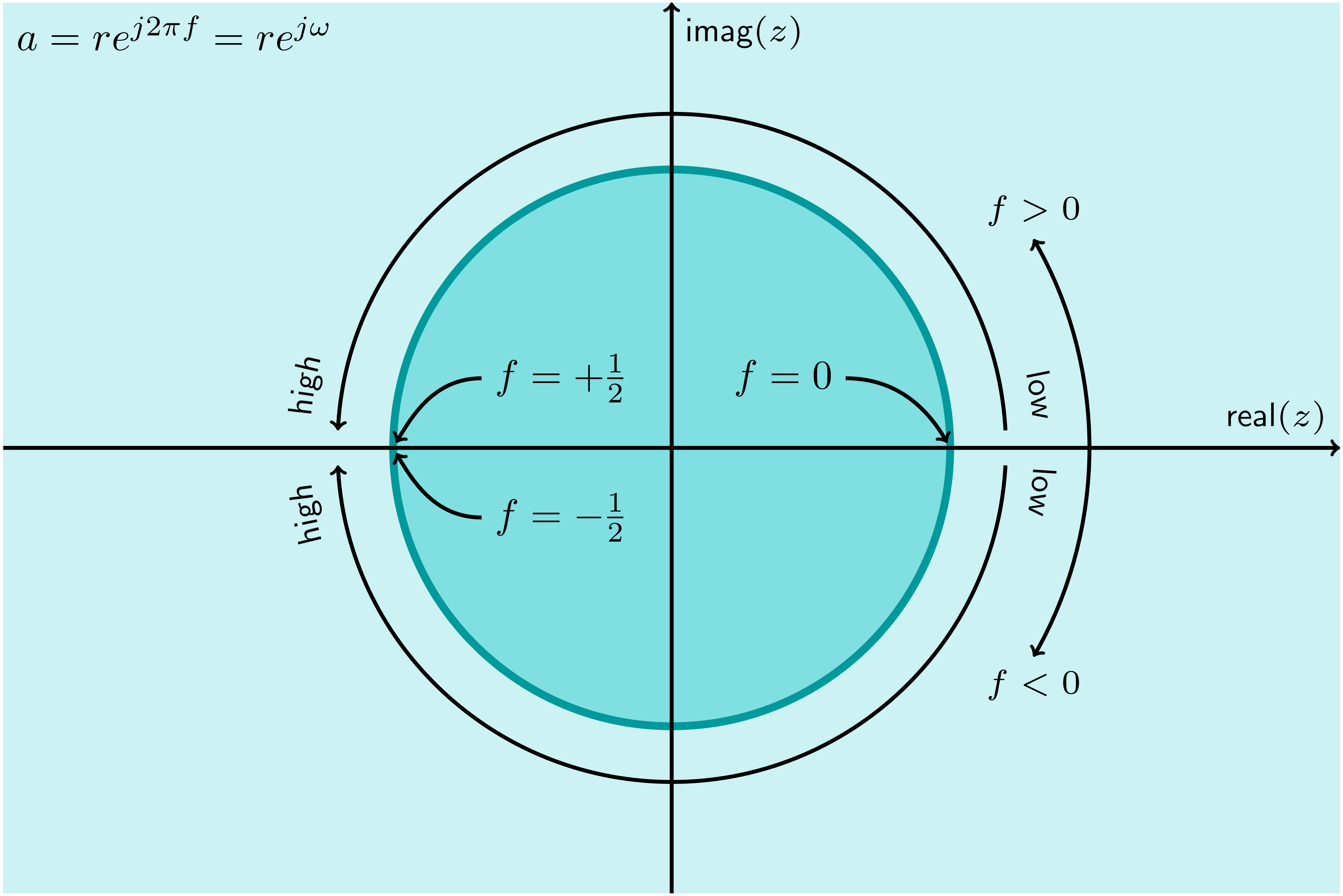
fundamental frequency interval $0 \le f < 1$



frequency aliasing



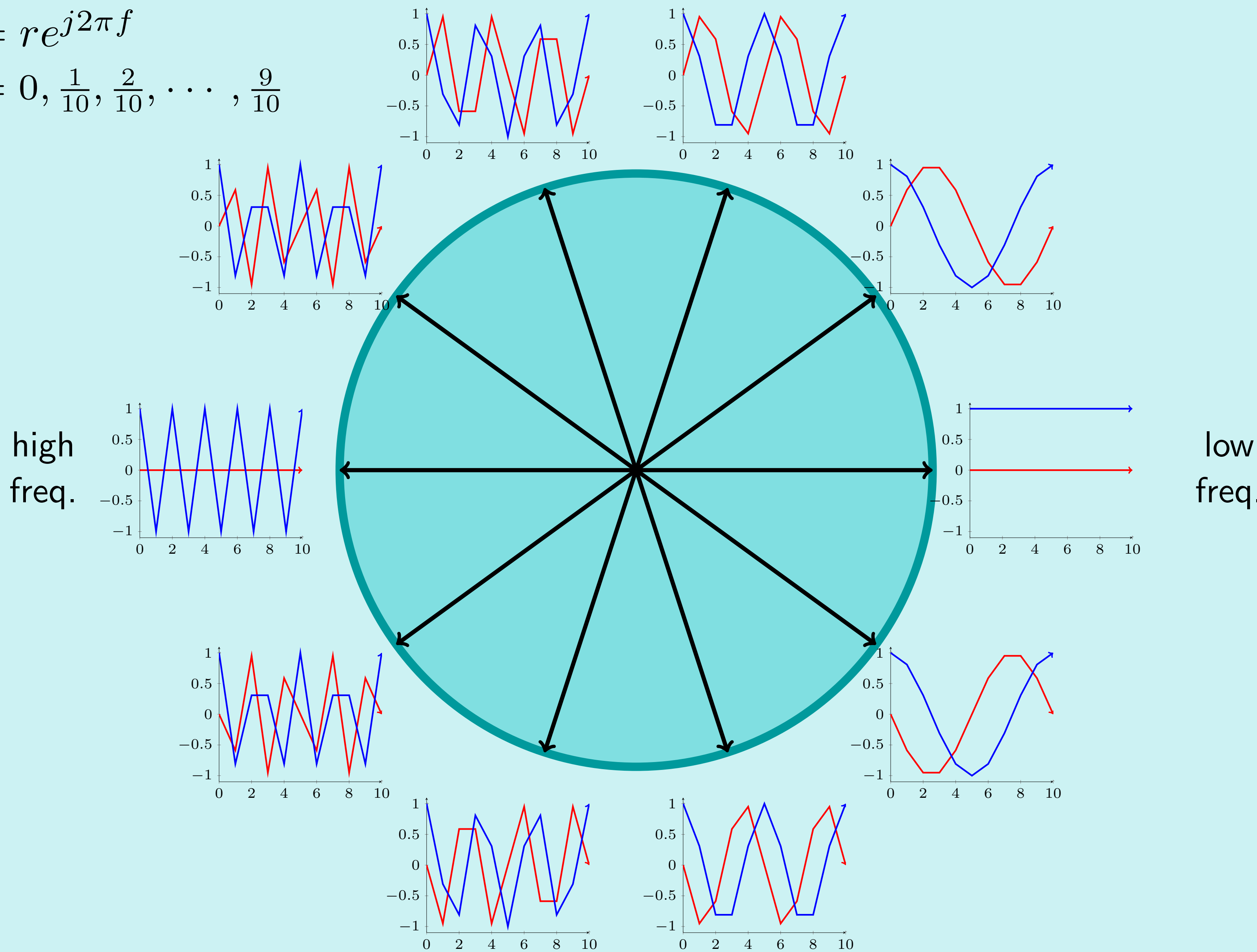
frequency of exponential $x[n] = a^n u[n]$



frequency of DT CE

$$a = re^{j2\pi f}$$

$$f = 0, \tfrac{1}{10}, \tfrac{2}{10}, \cdots, \tfrac{9}{10}$$



oscillatory behavior of CT CE

$$e^{j2\pi Ft}$$

rate of oscillation increases indefinitely as $F \to \infty$ or as $T = 1/F \to 0$.

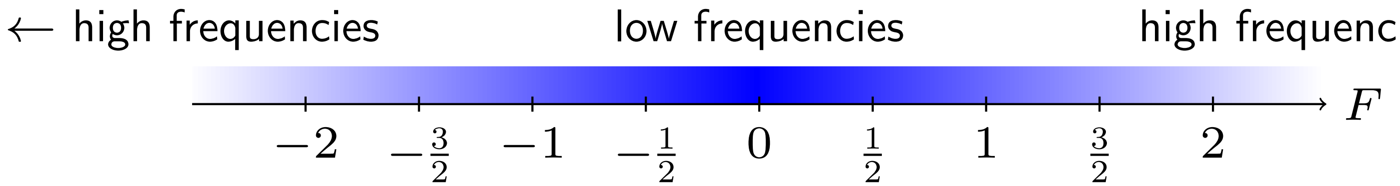
oscillatory behavior of DT CE

$$e^{j2\pi fn}$$

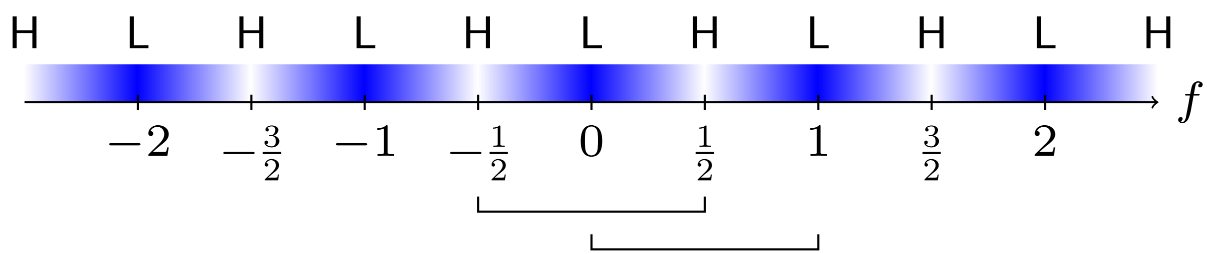
- rate of oscillation increases on $0 \le f < \frac{1}{2}$ and decreases on $\frac{1}{2} \le f < 1$.
- "low" frequencies near $0, \pm 1, \pm 2, \cdots$
- "high" frequencies near $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$
- $f = 0$ is the lowest frequency
- $f = \frac{1}{2} = -\frac{1}{2}$ is the highest frequency

CT and DT frequency

CT frequency



DT frequency



phase shifts and time shifts

$$e^{j2\pi F(t-\tau)} = e^{j[2\pi Ft - \varphi]}, \qquad \varphi = 2\pi F\tau \ \text{[radians]}$$

$$e^{j2\pi f(n-m)} = e^{j[2\pi fn - \varphi]}, \qquad \varphi = 2\pi fm \ \text{[radians]}$$

- CT: For every phase shift, there is a corresponding time shift of the waveform.
- DT: For some phase shifts, there is a corresponding time shift of the sequence.

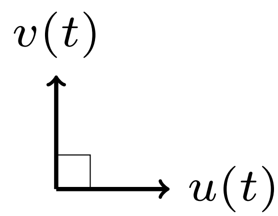
geometry of CT signals

inner product and norm (length) of CT signals

$$\langle u(t), v(t) \rangle = \int u(\tau) v^*(\tau) d\tau = \|u(t)\| \cdot \|v(t)\| \cos\theta$$

$$\|u(t)\| = \left[\int |u(\tau)|^2 d\tau \right]^{\frac{1}{2}} = \langle u(t), u(t) \rangle^{\frac{1}{2}}$$

When $\langle u(t), v(t) \rangle = 0$, then $\theta = 90$ degrees and $u(t)$ and $v(t)$ are orthogonal.



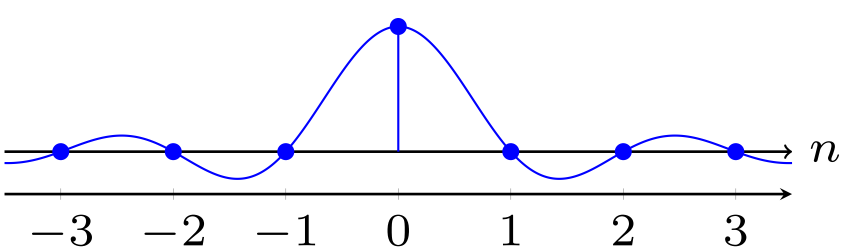
harmonically related CT CE signals are orthogonal

$$e^{j2\pi kFt}, \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

are harmonically related CE signals

Given F and $T = 1/F$, measure the angle between kth and mth harmonics:

$$\begin{aligned}
\langle e^{j2\pi kFt}, e^{j2\pi mFt} \rangle &= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi(k-m)Ft} dt \\
&= \frac{e^{\pi(k-m)} - e^{-\pi(k-m)}}{F \;\; 2j \;\; \pi(k-m)} \\
&= T \frac{\sin(\pi(k-m))}{\pi(k-m)} \\
&= T\delta[k-m] \\
&= \begin{cases} T, & k = m, \\ 0, & k \neq m \end{cases}
\end{aligned}$$



There are an infinite number of orthogonal harmonically related CT CE signals.

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi \frac{kt}{T}}, \quad F = \frac{1}{T} \qquad \text{(synthesis)}$$

$$X_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi \frac{kt}{T}} = \frac{1}{T} \left\langle x(t), e^{j2\pi \frac{kt}{T}} \right\rangle \qquad \text{(analysis)}$$

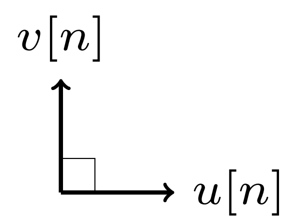
geometry of DT signals

inner product and norm (length) of DT signals

$$\langle u(t), v(t) \rangle = \sum_n u[n]v^*[n] = \|u[n]\| \cdot \|v[n]\| \cos\theta$$

$$\|u[n]\| = \left[\sum_n |u[n]|^2 \right]^{\frac{1}{2}} = \langle u[n], u[n] \rangle^{\frac{1}{2}}$$

When $\langle u[n], v[n] \rangle = 0$, then $\theta = 90$ degrees and $u[n]$ and $v[n]$ are orthogonal.



harmonically related DT CE signals are orthogonal

$$e^{j2\pi kfn}, \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

are harmonically related DT CE signals

Given $f = 1/N$ where N is the period, measure the angle between kth and mth harmonics:

$$\langle e^{j2\pi kfn}, e^{j2\pi mfn} \rangle = \sum_{n=0}^{N-1} e^{j2\pi(k-m)fn} = \begin{cases} N, & k = m + lN, l \in \mathbb{Z} \\ \frac{1-e^{j2\pi(k-m)}}{1-e^{j2\pi(k-m)f}} & \text{otherwise} \end{cases}$$

Consider the case $k \neq m + lN$,

$$\begin{aligned} \frac{1-e^{j2\pi(k-m)}}{1-e^{j2\pi(k-m)f}} &= \frac{e^{\pi(k-m)} - e^{-j\pi(k-m)}}{e^{j\pi(k-m)f} - e^{-j\pi(k-m)f}} e^{-j\pi(k-m)f(N-1)} \\ &= \frac{\sin(\pi(k-m))}{\sin(\pi(k-m)/N)} e^{-j\pi(k-m)(N-1)/N} = 0 \end{aligned}$$

There are only N orthogonal harmonically related DT CE signals.

DTFS

$$e^{j2\pi\frac{kn}{N}}, \quad k = 0, 1, 2, \cdots, N-1$$

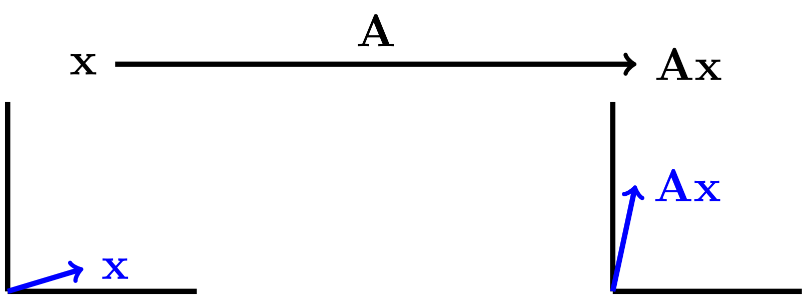
are a set of N orthogonal harmonically related DT CE signals.

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi\frac{kn}{N}}, \quad f = \frac{1}{N} \qquad \text{(synthesis)}$$

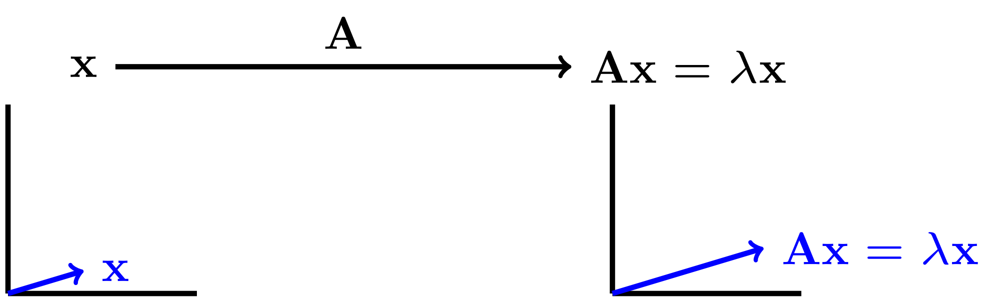
$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}} = \frac{1}{N}\left\langle x[n], e^{j2\pi\frac{kn}{N}}\right\rangle \qquad \text{(analysis)}$$

eigenvector of linear transformation

general vector $\mathbf{x}$: scale and direction change



eigenvector $\mathbf{x}$: scale changes, but direction stays the same



everlasting CT CE are eigenfunctions of CT LTI systems

$$x(t) = e^{j2\pi Ft} \longrightarrow \boxed{H(F)} \longrightarrow y(t) = H(F)e^{j2\pi Ft} = H(F)x(t)$$

$$H(F) = |H(F)|e^{j\angle H(F)}$$

$$y(t) = h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau = \underbrace{\int h(\tau)e^{-j2\pi F\tau}d\tau}_{\text{CTFT of } h(t)} \; e^{j2\pi Ft} = H(F)x(t)$$

- CT CE signal passes through CT LTI system with only a complex scale factor (magnitude and phase adjustment)

- CT CE is an eigenfunction of CT LTI system

- $H(F) = \int h(t)\exp(-j2\pi Ft)dt$ called frequency response of the system

everlasting DT CE are eigenfunctions of DT LTI systems

$$x[n] = e^{j2\pi fn} \longrightarrow \boxed{H(f)} \longrightarrow y[n] = H(f)e^{j2\pi fn} = H(f)x[n]$$

$$H(f) = |H(f)|e^{j\angle H(f)}$$

$$y[n] = h[n] * x[n] = \sum_k h[k]x[n-k] = \underbrace{\sum_k h[k]e^{-j2\pi fk}}_{\text{DTFT of } h[n]} \; e^{j2\pi fn} = H(f)x[n]$$

- DT CE signal passes through DT LTI system with only a complex scale factor (magnitude and phase adjustment)

- DT CE is an eigenfunction of DT LTI system

- $H(f) = \sum_n h[n]\exp(-j2\pi fn)$ called frequency response of the system

summary

property	CT CE	DT CE
frequency	F [Hz] Ω [rads/sec]	f [cycles/sample] ω [radians/sample]
periodicity	periodic for all F period $= 1/F$	periodic for rational $f = k/N$ period $= N$
uniqueness	distinct when frequencies are distinct	aliasing: f and $f + k$ give same sequence
oscillation	rate increases indefinitely with F	frequency axis alternates between high and low frequencies
phase shifts	there is time shift for every phase shift	there are time shifts for some phase shifts
number orthogonal harmonically related CE signals	∞	N
eigenfunction of LTI system	yes	yes
frequency response	$H(F) = \int h(t)e^{-j2\pi Ft}dt$	$H(F) = \sum_n h[n]e^{-j2\pi fn}$

assignment

1. Define and sketch the three types of DT CE signals: everlasting, causal, and finite (windowed).

2. When a CT CE with frequency $F = 440$ Hz is sampled at $\frac{1}{T} = 8000$ samples/second, what is the frequency f of the resulting DT CE signal?

3. If a CT CE is reconstructed from DT CE with frequency $f = 0.26257$ cycles/sample using a sample rate of $\frac{1}{T} = 6$ Giga samples/second, what is the resulting frequency F in Hertz?

4. What is the angular frequency ω associated with the cyclic frequency $f = 0.26257$ cycles/sample?

5. Let $x(t) = e^{jt}$ and $x[n] = e^{jn}$.

   (a) Explain why $x(t)$ is periodic but $x[n]$ is not.
   (b) What are the frequencies of $x(t)$ and $x[n]$?
   (c) What is the period of $x(t)$?

6. Let $x[n] = e^{j2\pi fn}$ where $f = \frac{213}{355}$.

   (a) Explain why $x[n]$ is periodic.
   (b) What is the period of $x[n]$?

7. Do the following in Matlab.

   (a) Plot $e^{j2\pi 0.1t}$ and $e^{j2\pi 1.1t}$ for $0 \leq t \leq 10$ on the same axis. (Hint: Use `t=[0:0.01:10];` to generate the time samples.)
   (b) On the same axis add $e^{j2\pi 0.1n}$ and $e^{j2\pi 1.1n}$ as stem plots. (Hint: Use Matlab's `stem` function instead of the `plot` function. Use `n=[0:10];` to generate the time samples.)
   (c) Explain why $F = 0.1$ Hz and $F = 1.1$ Hz give different CT CE signals while $f = 0.1$ cycles/sample and $f = 1.1$ cycles/sample gave the same DT CE sequence.
   (d) Draw the unit circle on the complex plane. Show the point $e^{j2\pi 0.1} = e^{j2\pi 1.1}$ and use the fact $(e^{j2\pi 0.1})^n = e^{j2\pi 0.1n} = (e^{j2\pi 1.1})^n = e^{j2\pi 1.1n}$ to explain frequency aliasing which is that $e^{j2\pi fn} = e^{j2\pi(f+k)n}$ for all n, where $k \in \mathbb{Z}$.

8. Find a frequency alias for $f = 37.8$ cycles/sample in the fundamental interval assuming:

   (a) $0 \leq f < 1$ is the fundamental interval
   (b) $-\frac{1}{2} \leq f < \frac{1}{2}$ is the fundamental interval

9. Why is $f = \pm\frac{1}{2}$ cycles/sample the highest frequency in discrete time?

10. Which is a higher frequency $f = 26.4$ cycles/sample or 38.9 cycles/sample? (Hint: Compare their aliased frequencies.)

11. Explain why $\frac{\sin(\pi n)}{\pi n} = \delta[n]$. Include a sketch in your explanation.

12. Prove by integration that $e^{j2\pi 10t}$ and $e^{j2\pi 30t}$ are orthogonal over the time interval $0 \leq t < 0.2$ seconds.

13. Prove by summation that $e^{j\frac{2\pi 2n}{9}}$ and $e^{j\frac{2\pi 4n}{9}}$ are orthogonal over $0 \leq n < 9$.

14. Suppose the everlasting DT CE sequence

$$x[n] = e^{j2\pi fn}, \quad -\infty < n < \infty$$

is applied to a DT LTI system with frequency response

$$H(f) = \frac{\sin(3\pi f)}{\sin(\pi f)} e^{-j2\pi f}.$$

What is the resulting output signal $y[n]$ if

(a) $f = \frac{2}{3}$
(b) $f = \frac{1}{2}$